Solutions for a selected set of problems from the text
"Mathematical Tools for Physics" by James Nearing

1.7 Factor the numerator of $\sinh 2y$, it is the difference of squares:

$$\sinh 2y = \frac{e^{2y} - e^{-2y}}{2} = \frac{\big(e^y + e^{-y}\big)\big(e^y - e^{-y}\big)}{2} = 2\cosh y \sinh y$$

$$\cosh 2y = \frac{e^{2y} + e^{-2y}}{2} = \frac{e^{2y} + 2 + e^{-2y} - 2}{2} = \frac{\big(e^y + e^{-y}\big)^2 - 2}{2} = 2\cosh^2 y - 1$$

1.11 Neither of these integrals make any sense. Both are divergent at $t = 0$. It is surprising however how often students who do this problem will come up with a number for the result. Some will argue that the integrand with $n = 1$ is odd and so its integral is zero — this shows some thinking about the problem, but it is not a principal value. For $n = 2$ some will manipulate it and come up with an answer that is not only finite, but negative — I'm not sure how, because the integrand is positive everywhere.

1.13

$$\frac{d}{d\alpha}\int_0^x dt\, e^{-\alpha t^2} = -\int_0^x dt\, t^2 e^{-\alpha t^2}$$

You can also change variables in the integral, letting $\alpha t^2 = u^2$.

$$\frac{d}{d\alpha}\int_0^{\sqrt{\alpha}x} \frac{du}{\sqrt{\alpha}} e^{-u^2} = \frac{1}{2\sqrt{\alpha}} x \frac{1}{\sqrt{\alpha}} e^{-\alpha x^2} + \int_0^{\sqrt{\alpha}x} \frac{-du}{2\alpha^{3/2}} e^{-u^2}$$

Set $\alpha = 1$; use the definition of erf, and you have the desired identity. See also problem 1.47.

1.18 Differentiate $\Gamma(x+1) = x\Gamma(x)$, to get $\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x)$. Apply this to $x = 1$, $x = 2$ *etc.* , and

$$\Gamma'(2) = \Gamma(1) + \Gamma'(1) = 1 - \gamma, \qquad \Gamma'(3) = \Gamma(2) + 2\Gamma'(2) = 1 + 2(1-\gamma) = 3 - 2\gamma$$

and all the other integers follow the same way.

1.19 Start from $\Gamma(1/2) = \sqrt{\pi}$ and the identity $x\Gamma(x) = \Gamma(x+1)$. If you are at the value $x = 1/2$, then you simply multiply the value of the Γ-function successively by $1/2$, $3/2$, ..., (n-$1/2$) to get

$$\Gamma(n + 1/2) = \sqrt{\pi}\,\frac{1}{2}\frac{3}{2}\frac{5}{2}\cdots\frac{2n-1}{2} = \sqrt{\pi}\,\frac{(2n-1)!!}{2^n}$$

1.20 Let $t^a = u$, then this is $\frac{1}{a}\Gamma(1/a)$.

1.21

$$c^2 = a^2 + b^2 - 2ab\cos\gamma, \qquad A = \tfrac{1}{2}\, ab\sin\gamma$$

Rearrange this, square, and add.

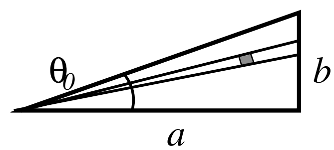
$$c^2 - a^2 - b^2 = -2ab\cos\gamma, \qquad 4A = 2ab\sin\gamma \quad\to\quad (c^2 - a^2 - b^2)^2 + 16A^2 = 4a^2b^2$$

$$\begin{aligned}16A^2 &= 4a^2b^2 - (c^2 - a^2 - b^2)^2 = (2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2)\\ &= ((a+b)^2 - c^2)(c^2 - (a-b)^2)) = (a+b+c)(a+b-c)(c+a-b)(c-a+c)\\ &= (a+b+c)(a+b+c-2c)(a+b+c-2b)(a+b+c-2a)\\ &= (2s)(2s-2c)(2s-2b)(2s-2a)\end{aligned}$$

where $s = (a+b+c)/2$ is the semiperimeter of the triangle, and the square root of this is the result: $A = \sqrt{s(s-a)(s-b)(s-c)}$.

1.27 Let θ_0 be the maximum angle: $\tan\theta_0 = b/a$.

$$\begin{aligned}\int_0^{\theta_0} d\theta \int_0^{a/\cos\theta} r\,dr &= \int_0^{\theta_0} d\theta \frac{1}{2}\left(\frac{a}{\cos\theta}\right)^2 = \frac{a^2}{2}\int_0^{\theta_0} d\theta\,\sec^2\theta\\ &= \frac{a^2}{2}\tan\theta_0 = \frac{a^2}{2}\frac{b}{a} = \frac{ab}{2}\end{aligned}$$



Doing the polar integral in the reverse order is much harder.

1.28 The chain rule is

$$\begin{aligned}\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{g\big(h(x+\Delta x)\big) - g\big(h(x)\big)}{\Delta x}\\ &= \frac{g\big(h(x+\Delta x)\big) - g\big(h(x)\big)}{h(x+\Delta x) - h(x)} \cdot \frac{h(x+\Delta x) - h(x)}{\Delta x} \longrightarrow \frac{dg}{dh}\frac{dh}{dx}\end{aligned}$$

2.1 For a loan L, the sum to compute the monthly payments is

$$L(1+i)^N - p\Big[(1+i)^{N-1} + (1+i)^{N-2} + \cdots + 1\Big] = 0$$

and this is

$$L(1+i)^N = p\sum_0^{N-1}(1+i)^k = p\frac{1-(1+i)^N}{1-(1+i)} \qquad \text{so} \qquad p = \frac{iL(1+i)^N}{(1+i)^N - 1}$$

Check: If $N = 1$, this is $iL(1+i)/\big[(1+i) - 1\big] = L(1+i)$.
Check: For $i \to 0$, this is

$$p \to \frac{iL(1+Ni)}{1+Ni-1} = \frac{L(1+Ni)}{N} \to \frac{L}{N}$$

The numerical result is ($i = .06/12 = .005$ and $N = 12 \times 30$)

$$\$200,000 \times \frac{.005(1+.005)^{360}}{(1+.005)^{360} - 1} = \$1199.10$$

The total paid over 30 years is then \$431676. This ignores the change in the value of money.

2.2 Let the monthly inflation rate be $r = 1.02^{1/12}$. The nominal amount paid over 30 years is just the \$431676 from the preceding problem. In constant dollars, the quantity paid is

$$\frac{p}{r} + \frac{p}{r^2} + \cdots + \frac{p}{r^N} = \frac{p}{r}\frac{1-r^{-N}}{1-1/r} = p\frac{1-r^{-N}}{r-1} = \$1199.10 \times 271.2123 = \$325211$$

2.29 Consider the series

$$f(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots$$

This is the derivative of

$$F(x) = \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots$$

Notice that $xF(x)$ is almost e^x. In fact $xF(x) = e^x - x - 1$. Solve for F and differentiate.

$$F(x) = \frac{1}{x}e^x - 1 - \frac{1}{x}, \qquad \text{so} \qquad f(x) = F'(x) = \frac{-1}{x^2}e^x + \frac{1}{x}e^x + \frac{1}{x^2}$$

Evaluate this at $x = 1$ to get 1.

2.31 To the lowest order in the speed, these three expressions are all the same.

$$f' = f\big(1 - v_o/v\big), \qquad\qquad f' = f\big(1 - v_s/v\big), \qquad\qquad f' = f\big(1 - v/c\big)$$

2.33 The depth of the object should appear shallower than in the absence of the medium. Answers (1), (3), and (5) do the opposite. For (5) this statement holds only for large n. If $n = 1$ the result should simply be d, and numbers (1), (2), and (5) violate this. All that is left is (4).

2.35 The travel time is

$$\begin{aligned} T &= \frac{1}{c}\sqrt{(R\sin\theta)^2 + (p + R - R\cos\theta)^2} + \frac{n}{c}\sqrt{(R\sin\theta)^2 + (q - R + R\cos\theta)^2} \\ &= \frac{1}{c}\sqrt{R^2 + (p+R)^2 - 2R(p+R)\cos\theta} + \frac{n}{c}\sqrt{R^2 + (q-R)^2 + 2R(q-R)\cos\theta} \end{aligned}$$

Rewrite this for small θ, expanding the cosine to second order.

$$\begin{aligned} T &= \frac{1}{c}\sqrt{p^2 + R(p+R)\theta^2} + \frac{n}{c}\sqrt{q^2 - R(q-R)\theta^2} \\ &= \frac{1}{c}p\sqrt{1 + R(p+R)\theta^2/p^2} + \frac{n}{c}q\sqrt{1 - R(q-R)\theta^2/q^2} \\ &= \frac{1}{c}p\left[1 + R(p+R)\theta^2/2p^2\right] + \frac{n}{c}q\left[1 - R(q-R)\theta^2/2q^2\right] \\ cT &= p + nq + \frac{1}{2}\theta^2\left[\frac{R^2 + Rp}{p} + n\frac{R^2 - Rq}{q}\right] = p + nq + \frac{1}{2}\theta^2 R^2\left(\frac{1}{p} + \frac{n}{q} - \frac{n-1}{R}\right) \end{aligned}$$

If the coefficient of θ^2 is positive, this is a minimum. That will happen if the values of p and q are small enough. Otherwise it is a maximum. The transition occurs when

$$\frac{1}{p} + \frac{n}{q} = \frac{n-1}{R}$$

This is the condition for a focus. This phenomenon is quite general, and the principle of "least time" is least only up to the position of a focus. After that is is a saddle point. That is, maximum with respect to long wavelength variations such as this one, and minimum with respect to short wavelength wiggles.

2.36

$$\ln(\cos\theta) = \ln(1 - \theta^2/2 + \theta^4/24 - \cdots) \qquad \text{and} \qquad \ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

Let $x = -\theta^2/2 + \theta^4/24 - \cdots$, then this series is

$$\begin{aligned}&\left[-\frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} +\right] - \frac{1}{2}\left[-\frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} +\right]^2 + \frac{1}{3}\left[-\frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} +\right]^3 - \cdots \\ &= -\frac{\theta^2}{2} - \frac{\theta^4}{12} - \frac{\theta^6}{45}\cdots\end{aligned}$$

2.37

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \qquad \text{and} \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+x) - \ln(1-x) = 2x + {}^2\!/\!_3 x^3 + {}^2\!/\!_5 x^5 + \cdots$$

The last series has the same domain of convergence in x as do the two that made it up, however the corresponding argument of the logarithm, $(1+x)/(1-x)$, goes from 0 to ∞.

2.46

$$\sum_0^\infty (-1)^k t^{2k}, \qquad \sum_0^\infty (-1)^k t^{-2-2k}$$

The first converges for $|t| < 1$ and second for $|t| > 1$.

2.55 (a) The series for the log is $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$, expand this value of $y(t)$ for small time. Let $gt/v_t = \gamma$ and observe that there is no t in the denominator, so inside the logarithm, it is necessary to keep terms only to order t^2 in order to get a final result that is accurate to that order. First the sine and cosine expansions:

$$\begin{aligned}y(t) &\approx \frac{v_t^2}{g}\ln\left(\left(v_t[1 - \tfrac{1}{2}\gamma^2] + v_0\gamma\right)/v_t\right) \\ &= \frac{v_t^2}{g}\ln\left(1 - \tfrac{1}{2}\gamma^2 + v_0\gamma/v_t\right) \approx \frac{v_t^2}{g}\left(-\tfrac{1}{2}\gamma^2 + v_0\gamma/v_t - \tfrac{1}{2}v_0^2\gamma^2/v_t^2\right)\end{aligned}$$

Collect all the terms and then use the value of γ.

$$\begin{aligned}y(t) &\approx \frac{v_t^2}{g}\left(\frac{v_0}{v_t}\frac{gt}{v_t} - \frac{1}{2}\frac{g^2t^2}{v_t^2} - \frac{1}{2}\frac{v_0^2}{v_t^2}\frac{g^2t^2}{v_t^2}\right) \\ &= v_0t - \tfrac{1}{2}\left(g + \frac{v_0^2 g}{v_t^2}\right)t^2\end{aligned}$$

This is in the form $y = v_0 t + a_y t^2/2$, the elementary expression for constant acceleration. The initial velocity is right: v_0. The acceleration is

$$a_y = -g - \frac{g v_0^2}{v_t^2} \qquad \text{and} \qquad F_y = m a_y$$

The first term in $m a_y$ is the usual gravitational force $-mg$. The second is another negative term caused by air resistance. It is proportional to the square of the velocity v_0, and that is precisely the velocity that the mass has when it starts. To this order at least, the air resistance is proportional to the square of the velocity.

(b) The maximum height occurs when $v_y = 0$, so from the preceding problem that is when the numerator is zero.

$$v_0 - v_t \tan(gt/v_t) = 0 \longrightarrow \tan(gt/v_t) = v_0/v_t$$

Knowing the tangent, you have the cosine and sine to use in the $y(t)$ equation.

$$\cos = \frac{1}{\sqrt{1+\tan^2}}, \qquad \text{and} \qquad \sin = \frac{\tan}{\sqrt{1+\tan^2}}$$

$$y_{\text{max}} = \frac{v_t^2}{g}\ln\left(\frac{v_t + v_0\frac{v_0}{v_t}}{v_t\sqrt{1+v_0^2/v_t^2}}\right) = \frac{v_t^2}{g}\ln\left(\frac{v_t^2+v_0^2}{v_t\sqrt{v_t^2+v_0^2}}\right) = \frac{v_t^2}{2g}\ln\left(1+v_0^2/v_t^2\right)$$

If the initial speed is very small, $v_0 \ll v_t$, then this result is approximately

$$y_{\text{max}} \approx \frac{v_t^2}{2g}\cdot\frac{v_0^2}{v_t^2} = \frac{v_0^2}{2g}$$

This is the result you get from the elementary solution with no air resistance.
If instead you give a very large initial speed, $v_0 \gg v_t$, then $y_{\text{max}} \approx \left(v_t^2/g\right)\ln(v_0/v_t)$. It increases only very slowly with higher initial speeds. There's a curious point here in that the *time* to reach the maximum height is bounded. $\tan(gt/v_t) = v_0/v_t$ implies that as $v_0 \to \infty$, the quotient $gt/v_t \to \pi/2$. If $v_t = 100\,\text{m/s}$ this time is about $t = 16\,\text{s}$. If $v_0 = 400\,\text{m/s}$, this height is about 1.4 km.

3.10 The n^{th} roots of one are

$$e^{2\pi i k/n}, \quad k = 0, 1, \ldots (n-1), \qquad \text{so} \qquad \sum_{k=0}^{n-1} e^{2\pi i k/n} = \frac{1-\left(e^{2\pi i/n}\right)^n}{1-e^{2\pi i/n}}$$

The numerator of this sum is zero.

3.11 $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) = 0$. As e^x is never zero for real x, it requires $\cos y = 0$ and $\sin y = 0$ simultaneously. This violates the identity $\cos^2 y + \sin^2 y = 1$.

3.14 The equation for an ellipse involves some algebra:

$$
\begin{gathered}
|z-f|+|z+f|=2a \;\rightarrow\; |z-f|=2a-|z+f|, \quad \text{and square it:}\\
(z-f)(z^*-f)=4a^2+(z+f)(z^*+f)-4a|z+f|\\
-2f(z+z^*)-4a^2=-4a|z+f| \;\rightarrow\; fx+a^2=a|z+f|\\
\text{square it:}\;\; (fx+a^2)^2=a^2(z+f)(z^*+f)\\
\rightarrow \quad f^2x^2+2a^2fx+a^4=a^2(x^2+y^2+f^2+2fx)\\
a^4-a^2f^2=(a^2-f^2)x^2+a^2y^2 \quad\rightarrow\quad 1=\frac{x^2}{a^2}+\frac{y^2}{a^2-f^2}
\end{gathered}
$$

3.18

$$
\frac{1+i}{1-i}=\frac{(1+i)^2}{(1-i)(1+i)}=\frac{2i}{2}=i
$$

OR notice that the magnitude of this is one, the numerator is at an angle $\pi/4$, and the denominator is at an angle $-\pi/4$. That gives the same result, $e^{i\pi/2}$.
The magnitude of the second fraction is one. The numerator is at an angle of $\pi/3$ above the negative x-axis, or $\theta=2\pi/3$, and the denominator is at angle $\pi/3$ above the positive x-axis. That gives $e^{2\pi i/3}/e^{i\pi/3}=e^{i\pi/3}$.
The third fraction has a numerator $i^5+i^3=0$. Done.
The magnitude of the fourth number is $\left(2/\sqrt{2}\right)^2=2$. The angle for the numerator is $\pi/6$, and for the denominator it is $\pi/4$. The result is then $2\left(e^{i\pi/6-i\pi/4}\right)^2=2e^{-i\pi/6}$.

3.26 For velocity and acceleration, do a couple of derivatives.

$$
\frac{dz}{dt}=\frac{dx}{dt}+i\frac{dy}{dt}=\frac{d}{dt}re^{i\theta}=\frac{dr}{dt}e^{i\theta}+ir\frac{d\theta}{dt}e^{i\theta}
$$

$$
\begin{aligned}
\frac{d^2}{dt^2}re^{i\theta}&=\frac{d^2r}{dt^2}e^{i\theta}+2i\frac{dr}{dt}\frac{d\theta}{dt}e^{i\theta}+ir\frac{d^2\theta}{dt^2}e^{i\theta}-r\left(\frac{d\theta}{dt}\right)^2e^{i\theta}\\
&=e^{i\theta}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]+ie^{i\theta}\left[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right]
\end{aligned}
$$

Translating this into the language of vectors, $\hat{r}$ points away from the origin as does $e^{i\theta}$. The factor i rotates by 90°. This is

$$
\vec{a}=\hat{r}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]+\hat{\theta}\left[r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}\right]
$$

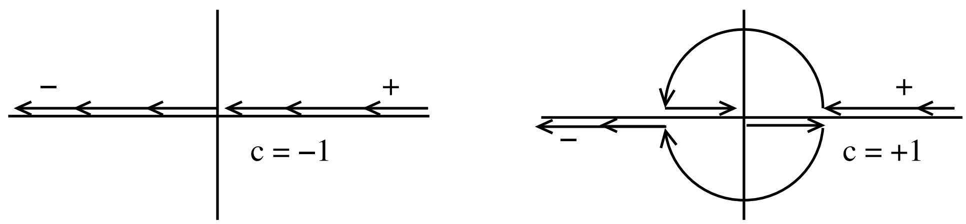
3.29 The quadratic equation is $z^2+bz+c=0$, then $z=(-b\pm\sqrt{b^2-4c})/2$ for the two cases $c=\pm1$ and over all real b.
For the case that $c=-1$, then $z=(-b\pm\sqrt{b^2+4})/2$ is always real.
For the case that $c=+1$, then $z=(-b\pm\sqrt{b^2-4})/2$ and z is real for $|b|>2$, or if $|b|<2$ it has magnitude $=1$, placing it on the unit circle around the origin.

		$c=-1$				$c=+1$				
Positive root:	$b:$	$-\infty$	0	$+\infty$	$b:$	$-\infty$	-2	0	$+2$	$+\infty$
	$z:$	$+\infty$	$+1$	0	$z:$	$+\infty$	$+1$	$+i$	-1	0
Negative root:	$b:$	$-\infty$	0	$+\infty$	$b:$	$-\infty$	-2	0	$+2$	$+\infty$
	$z:$	0	-1	$-\infty$	$b:$	0	$+1$	$-i$	-1	$-\infty$

Most of these cases are easy to find, but there are a few for which $b \to \pm\infty$ and which lead to the form $(\infty - \infty)$. In those cases write the root in the form

$$\frac{1}{2}\big(-b \pm |b|\sqrt{1 \pm 4/b^2}\big) = \frac{1}{2}\big(-b \pm |b|\big[1 \pm 2/b^2\big]\big)$$

and take the limit on b. The drawings show the paths of the paths that z takes in these four cases, with the labels "+" and "−" being the signs of the square roots.



3.44 $(2+i)(3+i) = 5+5i$. Now look at the polar form of the product, and the two angles on the left must add to the angle on the right: $\tan^{-1} {}^1\!/_2 + \tan^{-1} {}^1\!/_3 = \pi/4$.
For the next identity,

$$(5+i)^2 = 24+10i, \qquad (24+10i)^2 = 476+480i,$$
$$\text{then} \qquad (476+480i)(-239+i) = -114244 - 114244i$$

The angles again add, and $4\tan^{-1} {}^1\!/_5$ is one factor. The other is $\pi - \tan^{-1} {}^1\!/_{239}$. The right side has an angle $5\pi/4$.

$$4\tan^{-1} {}^1\!/_5 + \pi - \tan^{-1} {}^1\!/_{239} = 5\pi/4 \qquad \text{or} \qquad 4\tan^{-1} {}^1\!/_5 - \tan^{-1} {}^1\!/_{239} = \pi/4$$

To compute π to 100 places with an alternating series means that (barring special tricks) you want the n^{th} term to be less than 10^{-100}.

$$\frac{\pi}{4} = \tan^{-1} 1 = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} \qquad \text{then} \qquad 2k > 10^{100}, \qquad \text{or} \qquad k > \tfrac{1}{2}\text{Googol}$$

In the second series, the slower series is $\tan^{-1} {}^1\!/_2$, which is

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} \left(\frac{1}{2}\right)^{2k+1} \qquad \text{Now to the required accuracy} \qquad 2k \cdot 2^{2k+1} > 10^{100}$$

Take a logarithm: $(2k+1)\ln 2 + \ln 2k > 100 \ln 10$. The $\ln 2k$ term varies much more slowly than the other, so first ignore it

$$k > 50 \ln 100/\ln 2 = 332, \qquad \text{then improve it} \qquad k > (50\ln 100/\ln 2) - \tfrac{1}{2}\ln(2\cdot 332) = 329$$

The same calculation for $\tan^{-1} 1/3$ gives

$$k > 50 \ln 100/\ln 3 = 210 \qquad \text{improved to} \qquad k > (50 \ln 100/\ln 3) - \tfrac{1}{2}\ln(2\cdot 210) = 207$$

The total number of terms is $329 + 207 = 536$.
In the third series, the slower sum is $4\tan^{-1} 1/5$, with a sum

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1}\left(\frac{1}{5}\right)^{2k+1} \qquad \text{Now to the required accuracy} \qquad 2k\cdot 5^{2k+1} > 10^{100}$$

Again, take the logarithm: $(2k+1)\ln 5 + \ln 2k > 100 \ln 10$

$$k > 50\ln 100/\ln 5 = 143, \qquad \text{then improve it} \qquad k > (50\ln 100/\ln 2) - \tfrac{1}{2}\ln(2\cdot 143) = 140$$

The series for $\tan^{-1} 1/239$ takes $k > 50\ln 100/\ln 239 = 42$ more terms, totaling 182.

3.46 Combine the exponents and complete the square.

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\cos\beta x \to \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} e^{i\beta x} &= \int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x^2 - i\beta x/\alpha - \beta^2/4\alpha^2\right)} e^{-\beta^2/4\alpha^2} \\ &= e^{-\beta^2/4\alpha^2}\int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x - i\beta/2\alpha\right)^2} \\ &= e^{-\beta^2/4\alpha^2}\int_{-\infty}^{\infty} dx'\, e^{-\alpha x'^2} = e^{-\beta^2/4\alpha^2}\sqrt{\pi/\alpha}\end{aligned}$$

The final integration step involves pushing the contour from the real x-axis up to a parallel line along the contour through $+i\beta/2\alpha$. The other part of this complex integral, with the sine, is zero anyway because it's odd.

3.47 $\sin z = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y = 0$ requires both terms to vanish. $\cosh y$ is never zero for real y, so x must be a multiple of π. For such a value of x, the cosine is ± 1, and the only place the $\sinh$ vanishes is at $y = 0$. The familiar roots are then the only roots. The same argument applies to the cosine. For the tangent to vanish, either the sine is zero or the cosine is infinite. The latter doesn't happen, and the sine is already done.

3.49

$$\int_0^1 \frac{dx}{1+x^2} = \int_0^1 dx\,\frac{i}{2}\left[\frac{1}{x+i} - \frac{1}{x-i}\right] = \frac{i}{2}\Big[\ln(x+i) - \ln(x-i)\Big]_0^1 = \frac{i}{2}\left[\ln\frac{1+i}{i} - \ln\frac{1-i}{-i}\right]$$

The real parts of the two logarithms are the same, so they cancel: $\ln re^{i\theta} = \ln r + i\theta$. The angle going from i to $1+i$ is $-\pi/4$. The angle going from $-i$ to $1-i$ is $+\pi/4$. This integral is then

$$\frac{i}{2}\left[\frac{-i\pi}{4} - \frac{i\pi}{4}\right] = \frac{\pi}{4}$$

4.1 $m\, d^2x/dt^2 = -bx + kx$. Assume $x(t) = Ae^{\alpha t}$ then $mA\alpha^2 e^{\alpha t} = -bA\alpha e^{\alpha t} + kAe^{\alpha t}$. This implies

$$\alpha = \big(-b \pm \sqrt{b^2 + 4km}\big)/2m$$

The square root is always bigger than b, so the term with the plus sign will have a positive value of α and so a growing exponential solution.
Apply the initial conditions to $x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t}$.

$$x(0) = A + B = 0, \qquad v_x(0) = A\alpha_+ + B\alpha_- = v_0$$
$$\implies \qquad A = \frac{v_0}{\alpha_+ - \alpha_-} = \frac{mv_0}{\sqrt{b^2 + 4km}}, \quad B = -A$$

Combine these to get

$$x(t) = \frac{mv_0}{\sqrt{b^2 + 4km}}\big[e^{\alpha_+ t} - e^{\alpha_- t}\big] = \frac{2mv_0}{\sqrt{b^2 + 4km}} e^{-bt/2m} \sinh\left(\sqrt{b^2/4m + k/m}\, t\right)$$

This is the same as the equation (4.10) from the text for the stable case, except that the circular sine has become a hyperbolic sine. For small time the $\sinh$ is linear, so this is approximately

$$x(t) \approx \frac{2mv_0}{\sqrt{b^2 + 4km}}\left(\sqrt{b^2/4m + k/m}\, t\right) = v_0 t$$

For large time the $e^{\alpha_+ t}$ dominates, giving exponential growth.

4.2 For the anti-damped oscillator, $m\ddot{x} - b\dot{x} + kx = 0$. Try the exponential solution $x = e^{\alpha t}$ to get

$$m\alpha^2 - b\alpha + k = 0, \qquad \text{so} \qquad \alpha = \big[b \pm \sqrt{b^2 - 4km}\,\big]/2m$$

The second term, the square root, is necessarily smaller than the first if it is even real, so either both α's are real and positive or they are complex with a positive real part.

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

These terms both grow as positive exponentials for large time whether they oscillate or not. The given initial conditions are $x(0) = 0$ and $v_x(0) = v_0$. These are

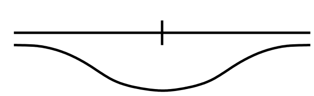
$$A_1 + A_2 = 0, \qquad A_1\alpha_1 + A_1\alpha_2 = v_0, \qquad \text{implying} \qquad A_1 = v_0/(\alpha_1 - \alpha_2), \qquad A_2 = -A_1$$

$$x(t) = \frac{2mv_0}{\sqrt{b^2 - 4km}} e^{bt/2m} \sinh\left(\sqrt{b^2 - 4km}\, t/2m\right)$$
$$\text{or} \qquad \frac{2mv_0}{\sqrt{4km - b^2}} e^{bt/2m} \sin\left(\sqrt{4km - b^2}\, t/2m\right)$$

The case $b^2 = 4km$ is a limit of either of these as $\sqrt{\ } \to 0$.

4.3 Near the origin,

$$V(x) = -V_0 \frac{a^2}{a^2 + x^2} = -V_0 \frac{1}{1 + x^2/a^2} = -V_0\big[1 - x^2/a^2 + \cdots\big]$$



If you don't make this approximation, the equation of motion is

$$m\ddot{x} = -dV/dx = -V_0 \frac{2a^2 x}{(a^2 + x^2)^2}$$

For small x this is $m\ddot{x} = -2V_0x/a^2$. That is so whether you is this equation or the approximate one for V. It is a harmonic oscillator with solution $e^{i\omega t}$, and $\omega^2 = 2V_0/ma^2$.
For the initial conditions $x(0) = x_0$ and $v_x(0) = 0$, use the cosine solution: $x(t) = x_0 \cos\omega t$.
As the parameter a gets very large, the function V becomes very deep and very wide. As the width gets large, the restoring force decreases, causing the oscillation frequency to decrease.

4.9 Starting from the result of the problem 4.8,

$$x(t) = \frac{F_0}{m} \cdot \frac{-\cos\omega_0 t + \cos\omega t}{(\omega_0 - \omega)(\omega_0 + \omega)}$$

As $\omega \to \omega_0$, the second factor in the denominator becomes $2\omega_0$. The rest of that quotient is the definition of the derivative of $-\cos\omega_0 t$ with respect to ω_0. The result is then

$$\frac{F_0}{2m\omega_0}\frac{d}{d\omega_0}(-\cos\omega_0 t) = \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

For small time, the series expansion of the sine says that this starts out as $(F_0/m)t^2/2$, which is the usual $at^2/2$ form for constant acceleration starting from rest. For large time the oscillations grow linearly.

4.10 Express everything in terms of $\cos\omega_0 t$ and $\sin\omega_0 t$. These are independent functions, so for this to be an identity their coefficients must match.

$$2(A + B) = C = E\cos\phi, \qquad 2i(A - B) = D = -E\sin\phi$$

From these, you get A and B in terms of C and D easily, and take the sum of squares for E.

$$C^2 + D^2 = E^2\cos^2\phi + E^2\sin^2\phi = E^2, \qquad \text{also divide} \qquad -D/C = \tan\phi$$

There are no constraints on any of these parameters, though you may get a surprise if for example $C = 1$ and $D = 2i$. Then $E = i\sqrt{3}$ and $\phi = -1.57 + 0.55i$.

4.17 This has an irregular singular point at $x = 0$, but assume $y = \sum_k a_k x^{k+s}$ anyway.

$$\sum_{k=0}^{\infty} a_k(k+s)(k+s-1)x^{k+s-2} + \sum_{k=0}^{\infty} a_k x^{k+s-3} = 0$$

Let $\ell = k$ in the first sum and $\ell = k - 1$ in the second.

$$\sum_{\ell=0}^{\infty} a_\ell(\ell+s)(\ell+s-1)x^{\ell+s-2} + \sum_{\ell=-1} a_{\ell+1}x^{\ell+s-2} = 0$$

The most singular term is in the second sum at $\ell = -1$. It is $a_0 x^{s-3}$. It has to equal zero all by itself and that contradicts the assumption that a_0 is the first non-vanishing term in the sum. The method fails.

4.18 For the equation $x^2u'' + 4xu' + (x^2+2)u = 0$ the Frobenius series solution, $u = \sum_0^\infty a_k x^{k+s}$ is

$$x^2 \sum_0^\infty a_k (k+s)(k+s-1)x^{k+s-2} + 4x \sum_0^\infty a_k (k+s)x^{k+s-1} + (x^2+2)\sum_0^\infty a_k x^{k+s} = 0$$

$$\sum_0^\infty a_k (k+s)(k+s-1)x^{k+s} + 4\sum_0^\infty a_k (k+s)x^{k+s} + 2\sum_0^\infty a_k x^{k+s} + \sum_0^\infty a_k x^{k+s+2} = 0$$

$$\sum_0^\infty a_k \big[(k+s)(k+s-1) + 4(k+s) + 2\big]x^{k+s} + \sum_0^\infty a_k x^{k+s+2} = 0$$

$$\sum_0^\infty a_k \big[(k+s)^2 + 3(k+s) + 2\big]x^{k+s} + \sum_0^\infty a_k x^{k+s+2} = 0$$

$$\sum_0^\infty a_k (k+s+1)(k+s+2)x^{k+s} + \sum_0^\infty a_k x^{k+s+2} = 0$$

$$\sum_{\ell=0}^\infty a_\ell (\ell+s+1)(\ell+s+2)x^{\ell+s} + \sum_{\ell=2}^\infty a_{\ell-2} x^{\ell+s} = 0$$

With the standard substitution, $\ell = k$ in the first sum and $\ell = k+2$ in the second. The most singular term comes from $\ell = 0$ in the first sum, and the recursion relation for a_ℓ comes from the rest.

$$a_0(s+1)(s+2) = 0 \qquad \text{and} \qquad a_\ell = -a_{\ell-2}\frac{1}{(\ell+s+1)(\ell+s+2)}$$

The $s = -2$ case gives

$$a_2 = -a_0\frac{1}{1\cdot 2}, \qquad a_4 = -a_2\frac{1}{3\cdot 4} = a_0\frac{1}{1\cdot 2\cdot 3\cdot 4}$$

The pattern is clear,

$$u = a_0\sum_{\ell=0}^\infty (-1)^\ell \frac{x^{2\ell-2}}{(2\ell)!} = a_0\frac{\cos x}{x^2}$$

For the other, $s = -1$ and

$$a_2 = -a_0\frac{1}{2\cdot 3}, \qquad a_4 = -a_2\frac{1}{4\cdot 5} = a_0\frac{1}{2\cdot 3\cdot 4\cdot 5},$$

$$\text{then} \qquad u = a_0\sum_{\ell=0}^\infty (-1)^\ell \frac{x^{2\ell-1}}{(2\ell+1)!} = \frac{\sin x}{x^2}$$

Where did I come up with this equation? I took the harmonic oscillator and made a substitution in order to turn it into a complicated looking equation.

4.20 $y'' + xy = 0$ and $y = \sum_0^\infty a_k x^{k+s}$, so

$$\sum_0^\infty a_k (k+s)(k+s-1)x^{k+s-2} + \sum_0^\infty a_k x^{k+s+1} = 0$$

Substitute $k-2=n$ in the first sum and $k+1=n$ in the second.

$$\sum_{n=-2}^{\infty} a_{n+2}(n+2+s)(n+s+1)x^{k+s} + \sum_{n=1}^{\infty} a_{n-1}x^{n+s} = 0$$

The indicial equation comes from $n=-2$ in the first sum: $a_0 s(s-1)=0$, so the possible values are $s=0,\ 1$. After that the recursion relation for the coefficients comes by setting the coefficient of x^{k+s} to zero.

$$a_{n+2}(n+2+s)(n+s+1) + a_{n-1} = 0,$$

$$\text{or, with } n=m+1 \quad a_{m+3} = -a_m \frac{1}{\big[(m+3+s)(m+2+s)\big]}$$

For the case $s=0$ this is

$$a_3 = -a_0\frac{1}{2\cdot 3}, \quad a_6 = -a_3\frac{1}{5\cdot 6} = +a_0\frac{4}{6!}, \quad a_9 = -a_6\frac{1}{8\cdot 9} = -a_0\frac{4\cdot 7}{9!}$$

$$y = 1 - \frac{x^3}{3!} + \frac{4\,x^6}{6!} - \frac{4\cdot 7\,x^9}{9!} + \cdots$$

For the $s=1$ case you have

$$a_3 = -a_0\frac{1}{3\cdot 4}, \quad a_6 = -a_3\frac{1}{6\cdot 7} = +a_0\frac{2\cdot 5}{7!}, \quad a_9 = -a_6\frac{1}{9\cdot 10} = -a_0\frac{2\cdot 5\cdot 8}{10!}$$

$$y = x - \frac{2\,x^4}{4!} + \frac{2\cdot 5\,x^7}{7!} - \frac{2\cdot 5\cdot 8\,x^{10}}{10!} + \cdots$$

4.25 $m\ddot{x} + kx = F_0\sin\omega_0 t$. The Green's function solution is, using Eq. (4.34),

$$x(t) = \frac{1}{m\omega_0}\int_0^t dt'\, F_0 \sin\omega_0(t')\sin\big(\omega_0(t-t')\big)$$

The trig identity for the product of two sines is $2\sin x\sin y = \cos(x-y) - \cos(x+y)$.

$$\begin{aligned} x(t) &= \frac{F_0}{2m\omega_0}\int_0^t dt'\,\big[\cos\omega_0(2t'-t) - \cos(\omega_0 t)\big] \\ &= \frac{F_0}{2m\omega_0}\left[\frac{1}{2\omega_0}\sin\omega_0(2t'-t) - t'\cos(\omega_0 t)\right]_0^t = \frac{F_0}{2m\omega_0^2}\left[\sin(\omega_0 t) - \omega_0 t\cos(\omega_0 t)\right] \end{aligned}$$

For small t, use series expansions.

$$x(t) \approx \frac{F_0}{2m\omega_0^2}\big[\omega_0 t - \omega_0^3 t^3/6 + \cdots - \omega_0 t(1 - \omega_0^2 t^2/2 + \cdots)\big] = \frac{F_0\omega_0}{2m}\big[t^3/3 + \cdots\big]$$

For a comparison, go back to the original differential equation. For small time, the position hasn't changed much from the origin, so $m\ddot{x} \approx F_0\omega_0 t$. Integrate this twice and use the initial conditions $x(0) = 0 = \dot{x}(0)$. You get $x(t) = F_0\omega_0 t^3/6m$.
For large time, the dominant term is the second: $x(t) \approx -F_0 t\cos(\omega_0 t)/2m\omega_0$. It grows without bound because the force is exactly at resonance and there's no damping.

4.28

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \text{and} \qquad \frac{dN_2}{dt} = -\lambda_2 N_2 + \lambda_1 N_1$$

The first equation has an exponential solution, $N_1 = N_0 e^{-\lambda_1 t}$. Put that into the second equation.

$$\frac{dN_2}{dt} = -\lambda_2 N_2 + \lambda_1 N_0 e^{-\lambda_1 t}$$

The homogeneous part (N_2) again has an exponential solution: $Ae^{-\lambda_2 t}$. For a solution of the inhomogeneous equation try a solution $N_2 = Ce^{-\lambda_1 t}$ and plug in.

$$-C\lambda_1 e^{-\lambda_1 t} + \lambda_2 C e^{-\lambda_1 t} = \lambda_1 N_0 e^{-\lambda_1 t}$$

This determines $C = \lambda_1 N_0 / \big(\lambda_2 - \lambda_1\big)$. The total solution is then

$$N_2 = Ae^{-\lambda_2 t} + \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}$$

The initial condition that $N_2(0) = 0$ determines A.

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

The total activity is the sum of the activities from elements #1 and #2: $\lambda_1 N_1 + \lambda_2 N_2$.

$$= \lambda_1 N_0 e^{-\lambda_1 t} + \lambda_2 \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right] = N_0 \lambda_1 \left[(2\lambda_2 - \lambda_1) e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}\right] / (\lambda_2 - \lambda_1)$$

As a check, if $\lambda_2 \to 0$, this reduces to the activity of the first element alone.
If $\lambda_2 \gg \lambda_1$, the second exponential disappears quickly and the result is $2\lambda_1 N_0 e^{-\lambda_1 t}$. That is double the activity of the single element. The initial activity is only $\lambda_1 N_0$, so it grows over time as the daughter grows. The factor of two appears because an equilibrium occurs after a long time, and for every parent that decays a daughter decays too.

4.37 The sequence of equations you get by differentiating the original equation determine all the higher derivatives. $x(0) = 0$ and $\dot{x}(0) = v_0$.

$$\begin{aligned}
\frac{d^2x}{dt^2} &= -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x && \text{at 0:} \quad \ddot{x}(0) = -\frac{b}{m}v_0 \\
\dddot{x} &= -\frac{b}{m}\ddot{x} - \frac{k}{m}\dot{x} && \text{at 0:} \quad \dddot{x}(0) = -\frac{b}{m}\left(-\frac{b}{m}v_0\right) - \frac{k}{m}v_0 \\
\ddddot{x} &= -\frac{b}{m}\dddot{x} - \frac{k}{m}\ddot{x} && \text{at 0:} \quad \ddddot{x}(0) = -\frac{b}{m}\left(\frac{b^2}{m^2}v_0 - \frac{k}{m}v_0\right) - \frac{k}{m}\left(-\frac{b}{m}v_0\right)
\end{aligned}$$

The power series expansion of the solution is then

$$x(t) = v_0\left[t - \frac{b}{m}\frac{t^2}{2} + \left(\frac{b^2}{m^2} - \frac{k}{m}\right)\frac{t^3}{6} + \left(-\frac{b^3}{m^3} + \frac{2kb}{m^2}\right)\frac{t^4}{24} + \cdots\right]$$

Compare this to the expansion of the solutions in Eq. (4.10).

$$\frac{v_0}{\omega'}e^{-\gamma t}\sin\omega' t = \frac{v_0}{\omega'}\left[1-\gamma t+\gamma^2t^2/2-\gamma^3t^3/6+\cdots\right]\left[\omega' t-\omega'^3t^3/6+\cdots\right]$$
$$= v_0\left[t-\gamma t^2+\left(\frac{\gamma^2}{2}-\frac{\omega'^2}{6}\right)t^3+\left(-\frac{\gamma^3}{6}+\frac{\gamma\omega'^2}{6}\right)t^4+\cdots\right]$$

The values of these parameters are $\gamma = b/2m$ and $\omega' = \sqrt{(k/m)-(b^2/4m^2)}$, and with these values everything conspires to agree.

4.38 A force F_0 that acts for a very small time at t' changes the velocity by $\Delta v = F_0\Delta t'/m$. The position is from then on, $\Delta v\,(t-t')$. Add many of these contributions, each from a force $F_x(t')$ acting for time $\Delta t'$. The total position function is then the sum of each of these contributions.

$$x(t) = \int_{-\infty}^{t} dt'\,\frac{F_x(t')}{m}(t-t')$$

For the special case $F_x = F_0$ for $t>0$ this is

$$x(t) = \frac{F_0}{m}\int_0^t dt'(t-t') = \frac{F_0}{m}t^2/2$$

so at least it works in this case.
How can the single integral accomplish the work of two? Differentiate x to verify the general result, noting that the variable t appears in two places.

$$\frac{dx}{dt} = \frac{1}{m}F_x(t')(t-t')\bigg|_{t'=t}+\int_{-\infty}^{t} dt'\,F_x(t'), \quad \text{then} \quad \frac{d^2x}{dt^2} = \frac{1}{m}\frac{d}{dt}\int_{-\infty}^{t} dt'\,F_x(t') = \frac{1}{m}F_x(t)$$

4.58 To solve $x^2y''-2ixy'+(x^2+i-1)y=0$ assume a solution $y=\sum_k a_kx^{k+s}$.

$$x^2\sum_{k=0}^{\infty} a_k(k+s)(k+s-1)x^{k+s-2}$$
$$-2ix\sum_k a_k(k+s)x^{k+s-1}+(x^2+i-1)\sum_k a_kx^{k+s}=0$$
$$\sum_{k=0}^{\infty} a_kx^{k+s}\big[(k+s)(k+s-1)-2i(k+s)+i-1\big]+\sum_k a_kx^{k+s+2}=0$$
$$\sum_{\ell=0}^{\infty} a_\ell x^{\ell+s}\big[(\ell+s)(\ell+s-1)-2i(\ell+s)+i-1\big]+\sum_{\ell=2}^{\infty} a_{\ell-2}x^{\ell+s}=0$$

The indicial equation comes from the $\ell=0$ term in the first sum:

$$s(s-1)-2is+i-1=0=s^2-s(2i+1)+i-1=(s-i-1)(s-i)$$

The values of s are now i and $i+1$. The recursion relation for a_ℓ is

$$a_\ell x^{\ell+s}\big[(\ell+s)(\ell+s-1)-2i(\ell+s)+i-1)\big]+a_{\ell-2}=0$$
$$a_\ell = \frac{-a_{\ell-2}}{(\ell+s)(\ell+s-1)-2i(\ell+s)+i-1)} = \frac{-a_{\ell-2}}{\ell^2+\ell(2s-1-2i)+s(s-1)-2is+i-1}$$

and the last set of terms in the denominator add to zero because of the indicial equation.

$$a_\ell = \frac{-a_{\ell-2}}{\ell^2 + \ell(2s-1-2i)}$$

The indicial equations give two sequences.

$$s = i \;\to\; a_\ell = \frac{-a_{\ell-2}}{\ell(\ell-1)} \qquad\qquad s = i+1 \;\to\; a_\ell = \frac{-a_{\ell-2}}{\ell(\ell+1)}$$

Start with $s = i$.

$$a_2 = a_0\frac{-1}{1\cdot 2}, \qquad a_4 = \frac{-a_2}{3\cdot 4} = a_0\frac{1}{4!}$$

The pattern is already apparent.

$$\sum_{\ell=0}^{\infty} a_\ell x^{\ell+i} = a_0 x^i\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] = a_0 x^i \cos x$$

The other series is

$$\sum_{\ell=0}^{\infty} a_\ell x^{\ell+i} = a_0' x^{i+1}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right] = a_0' x^i \sin x$$

5.3 $f(x) = 1$.

5.4 On the interval $0 < x < L$ and with boundary conditions $u'(0) = 0 = u'(L)$, the orthogonal functions are $u_n(x) = \cos(n\pi x/L)$ for $n = 0,\ 1,\ 2, \ldots$.

$$x^2 = \sum a_n u_n \;\Rightarrow\; \langle u_n, x^2\rangle = a_n\langle u_n, u_n\rangle$$

$$\text{or} \qquad \int_0^L dx\,\cos(n\pi x/L)x^2 = a_n\int_0^L dx\,\cos^2(n\pi x/L)$$

The integral on the right is easy because the average value of $\cos^2$ over a period (or half-period) is 1/2, so for $n \geq 1$ the integral is $L/2$. For $n = 0$ it is L.

$$\int_0^L dx\,\cos\alpha x = \frac{1}{\alpha}\sin\alpha L,$$

then by differentiation with respect to α you have

$$-\int_0^L dx\,x^2\cos\alpha x = \frac{d^2}{d\alpha^2}\frac{1}{\alpha}\sin\alpha L = \frac{2}{\alpha^3}\sin\alpha L - \frac{2L}{\alpha^2}\cos\alpha L - \frac{L^2}{\alpha}\sin\alpha L$$

Evaluate this at $\alpha = n\pi/L$ and the sine terms vanish. For $n \geq 1$ you have

$$\int_0^L dx\,\cos(n\pi x/L)x^2 = \frac{2L^3}{n^2\pi^2}(-1)^n = a_n\frac{L}{2} \qquad \text{and for } n = 0 \text{ this is} \qquad \frac{L^3}{3} = a_0 L$$

$$\sum_{n=0}^{\infty} a_n u_n = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x/L)$$

Graphs of these partial sums follow Eq. (5.1).

5.5 The basis functions are $u_n(x) = \sin(n\pi x/L)$. The Fourier series for $f(x) = x$ on this interval is

$$f = \sum a_n u_n, \qquad \text{then} \qquad \langle u_n, f\rangle = a_n\langle u_n, u_n\rangle,$$
$$\text{or} \qquad \int_0^L dx\, \sin\Big(\frac{n\pi x}{L}\Big)\, x = a_n \int_0^L dx\, \sin^2\Big(\frac{n\pi x}{L}\Big)$$

On the right, the average of $\sin^2$ is 1/2, so the integral is $L/2$. On the left,

$$\int_0^L dx\, \cos\alpha x = \frac{1}{\alpha}\sin\alpha L,$$
$$\text{then} \qquad \frac{d}{d\alpha}\frac{1}{\alpha}\sin\alpha L = -\int_0^L dx\, x\sin\alpha x = \frac{L}{\alpha}\cos\alpha L - \frac{1}{\alpha^2}\sin\alpha L$$

Set $\alpha = n\pi/L$ and the integral is $(-1)^{n+1}L^2/n\pi$.

$$a_n = \frac{2}{L}(-1)^{n+1}\frac{L^2}{n\pi}, \qquad \text{so} \qquad x = \frac{2L}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin\Big(\frac{n\pi x}{L}\Big)$$

This has a slow convergence rate, as the terms go to zero only as $1/n$. As a quick check, the first term $(n = 1)$ starts as $+\sin\pi x/L$, not $-$. That's how I found my own sign error as I wrote this out.

5.6 For the same function x using the basis $\sin\big(n + 1/2\big)\pi x/L$ the setup repeats that of the preceding problem, then

$$\langle u_n, f\rangle = a_n\langle u_n, u_n\rangle = \int_0^L dx\, \sin\bigg(\frac{(n + 1/2)\pi x}{L}\bigg)\, x = a_n\int_0^L dx\, \sin^2\bigg(\frac{(n + 1/2)\pi x}{L}\bigg)$$

The integral on the right is the same as before, averaging the sine^2. On the left, the same parametric differentiation works, with only a change in the value of α to $(n + 1/2)n\pi/L$. The integral is

$$-\frac{L}{\alpha}\cos\alpha L + \frac{1}{\alpha^2}\sin\alpha L = \frac{L^2}{(n + 1/2)^2\pi^2}(-1)^n$$

$$x = \frac{8L}{\pi^2}\sum_0^\infty \frac{(-1)^n}{(2n+1)^2}\sin\bigg(\frac{(n + 1/2)\pi x}{L}\bigg)$$

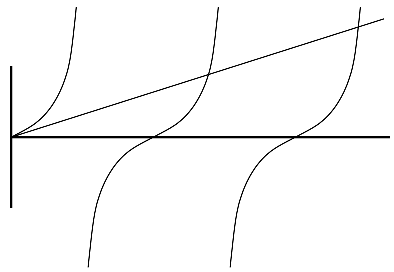
This converges more rapidly than the preceding case, going as $1/n^2$. Again, the first term starts as $+\sin\pi x/2L$ not $-$.

5.8 The boundary conditions proposed are $u(0) = 0$ and $u(L) = Lu'(L)$. Do these make the bilinear concomitant vanish?

$$u_1' u_2^* - u_1 u_2^{*\prime}\Big|_0^L = u_1'(L)\, L u_2^{*\prime}(L) - L u_1{}'(L)\, u_2^{*\prime}(L) - u_1'(0)\, u_2^*(0) + u_1(0)\, u_2{}'(0) = 0$$

This means that you can use the solutions with these boundary conditions for expansion functions. The condition at $x = 0$ is easy; that just means you're dealing with $\sin kx$. At the other limit,

$\sin kL = kL\cos kL$. This equation $\tan kL = kL$ has many solutions, as you can see from a quick sketch of the graph.



There's no neat analytic solution to this equation, but carry on. Call these solutions u_n, and as before,

$$f = \sum a_n u_n, \qquad \text{then} \qquad \langle u_n, f\rangle = a_n\langle u_n, u_n\rangle,$$

$$\text{or} \qquad \int_0^L dx\, x\sin k_n x = a_n \int_0^L dx\, \sin^2 k_n x$$

Plunge ahead and do these integrals.

$$\int_0^L dx\, x\sin kx = -\frac{d}{dk}\int_0^L dx\, \cos kx = -\frac{d}{dk}\frac{1}{k}\sin kL = \frac{1}{k^2}\sin kL - \frac{L}{k}\cos kL$$

Now use the equation that the k_n must satisfy, $\sin kL = kL\cos kL$. That gives

$$\frac{1}{k^2}\sin kL - \frac{L}{k}\cos kL = \frac{1}{k^2}kL\cos kL - \frac{L}{k}\cos kL = 0$$

This implies that the Fourier coefficient a_n vanishes for all values of n. The Fourier series vanishes identically. That's not supposed to happen!

How did this occur? It goes back to the discussion following the equation (5.16). The recommended procedure is to analyze all possible cases of the eigenvalue λ: positive, negative, and zero to determine which are allowed. That's the step that I skipped. There is a zero eigenvalue that is not a sine. It is x itself. That means that the complete Fourier series expansion in this basis is

$$x = x$$

5.10 The functions that vanish at $-\pi$ and π and that satisfy $u'' = \lambda u$ are $\sin n(x+\pi)/2$, $(n \geq 1)$.

$$\cos x = \sum_1^\infty a_n \sin n(x+\pi)/2, \qquad \text{so} \qquad \pi a_n = \int_{-\pi}^{\pi} dx\, \cos x \sin n(x+\pi)/2$$

The π comes because the average of $\sin^2$ is 1/2. The cosine is an even function, and the basis elements are odd if n is an even integer. The only ns that contribute are then odd. Use the trig identity

$$2\cos x\sin y = \sin(y+x) + \sin(y-x)$$

$$\begin{aligned}
2\pi a_n &= \int_{-\pi}^{\pi} dx\, \big[\sin\big((n+2)x/2 + n\pi/2\big) + \sin\big((n-2)x/2 + n\pi/2\big)\big] \\
&= \int_{-\pi}^{\pi} dx\, \big[\sin\big((n+2)x/2\big)\cos(n\pi/2) + \cos\big((n+2)x/2\big)\sin(n\pi/2) \\
&\qquad\qquad + \sin\big((n-2)x/2\big)\cos(n\pi/2) + \cos\big((n-2)x/2\big)\sin(n\pi/2)\big] \\
&= \int_{-\pi}^{\pi} dx\, \big[\cos\big((n+2)x/2\big)\sin(n\pi/2) + \cos\big((n-2)x/2\big)\sin(n\pi/2)\big] \\
&= \frac{2}{n+2}\sin\big((n+2)x/2\big)\sin(n\pi/2) + \frac{2}{n-2}\sin\big((n-2)x/2\big)\sin(n\pi/2)\bigg|_{-\pi}^{\pi}
\end{aligned}$$

Remember that n is odd when you go through these manipulations, in fact now looks like a good time to make it explicit: $n = 2k + 1$, $k \geq 0$.

$$\begin{aligned}
2\pi a_{2k+1} &= \frac{2}{2k+3}\sin\big((2k+3)x/2\big)\sin((2k+1)\pi/2) \\
&\qquad\qquad + \frac{2}{2k-1}\sin\big((2k-1)x/2\big) + \sin((2k+1)\pi/2)\bigg|_{-\pi}^{\pi} \\
&= \frac{2(-1)^k}{2k+3}\sin\big((2k+3)x/2\big) + \frac{2(-1)^k}{2k-1}\sin\big((2k-1)x/2\big)\bigg|_{-\pi}^{\pi}
\end{aligned}$$

The values at the lower limit duplicate those at the upper limit, making this

$$\begin{aligned}
2\pi a_{2k+1} &= \frac{4(-1)^k}{2k+3}(-1)^{k+1} + \frac{4(-1)^k}{2k-1}(-1)^{k+1} \\
a_{2k+1} &= -\frac{4}{\pi}\frac{2k+1}{(2k+3)(2k-1)}
\end{aligned}$$

This converges as $1/k$, and that is appropriate for this sum because all the sine terms vanish at the endpoints but the cosine doesn't. That causes slow convergence.

5.13 The basis functions are $u_n = e^{2n\pi it/T}$.

$$\big\langle u_n, e^{-\alpha t}\big\rangle = \big\langle u_n, \sum_m a_m u_m\big\rangle \qquad \text{is} \qquad \int_0^T dt\, e^{-2n\pi it/T}e^{-\alpha t} = a_n T$$

$$\begin{aligned}
a_n T &= \int_0^T dt\, e^{-(\alpha+2n\pi i/T)t} = \frac{-1}{\alpha + 2n\pi i/T}\Big[e^{-(\alpha+2n\pi i/T)T} - 1\Big] \\
&= \Big[1 - e^{-\alpha T}\Big]\frac{1}{\alpha + 2n\pi i/T}\cdot\frac{\alpha - 2n\pi i/T}{\alpha - 2n\pi i/T} \\
&= \Big[1 - e^{-\alpha T}\Big]\frac{\alpha - 2n\pi i/T}{\alpha^2 + 4n^2\pi^2/T^2} = \Big[1 - e^{-\alpha T}\Big]\frac{\alpha - ni\omega}{\alpha^2 + n^2\omega^2} \qquad\qquad (\omega = 2\pi/T) \\
e^{-\alpha t} &= \Big[\Big(1 - e^{-\alpha T}\Big)/T\Big]\sum_{-\infty}^{\infty} e^{ni\omega t}\frac{\alpha - ni\omega}{\alpha^2 + n^2\omega^2}
\end{aligned}$$

Now write this in terms of sines and cosines. The term in α contributes the cosine; the term in $ni\omega$ contributes the sine. That is because the cosine is even and the sine is odd.

$$e^{-\alpha t} = \Big[\Big(1 - e^{-\alpha T}\Big)/T\Big]\left[\frac{1}{\alpha} + 2\sum_1^{\infty}\frac{\alpha}{\alpha^2 + n^2\omega^2}\cos n\omega t + 2\sum_1^{\infty}\frac{n\omega}{\alpha^2 + n^2\omega^2}\sin n\omega t\right]$$

As $\alpha \to 0$, the combination $\big(1 - e^{-\alpha T}\big)$ is αT. The $1/\alpha$ term is all that's left in the sum and that combines with the overall coefficient to have the limit 1.
In the case of very large α, the cosine terms dominate. The sine terms have a $1/\alpha^2$ as coefficients. This looks like

$$2\sum_1^{\infty}\frac{\alpha T}{\alpha^2 T^2 + n^2\omega^2 T^2}\cos n\omega t$$

For large αT, the denominator changes very slowly as a function of n. This sum is approximately an integral.

$$2\int_0^\infty dn\,\frac{\alpha T}{\alpha^2T^2+n^2\omega^2T^2}\cos n\omega t$$

This integral can be done by the techniques of contour integration in chapter 14, or you can look it up in the table of integrals by Gradshteyn and Ryzhik: 3.723.2, where the result is $e^{-\alpha t}$.

5.22 Nothing.

$$f=\sum a_nu_n\ \to\ \langle u_n,f\rangle=a_n\langle u_n,u_n\rangle\ \to\ f=\sum\frac{\langle u_n,f\rangle}{\langle u_n,u_n\rangle}u_n$$

There are just as many factors of u_n in the numerator as in the denominator, and just as many complex conjugations, so multiplying the basis by any (complex) number changes nothing. Even scaling $u_n\to\alpha_nu_n$ with each element of the basis changed by a different factor has no effect.

5.31

$$\mathrm{Si}(x)=\frac{2}{\pi}\int_0^x dt\,\frac{\sin t}{t}=\frac{2}{\pi}\int_0^x dt\sum_0^\infty(-1)^n\frac{t^{2n}}{(2n+1)!}=\frac{2}{\pi}\sum_0^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)(2n+1)!}$$

5.35 x^4 is even, so I may as well use cosines over this interval.

$$x^4=\sum_0^\infty a_n\cos n\pi x/L\qquad\Rightarrow\qquad\big\langle\cos n\pi x/L,x^4\big\rangle=a_n\big\langle\cos n\pi x/L,\cos n\pi x/L\big\rangle$$

$$La_n=\int_{-L}^{L}dx\,x^4\cos n\pi x/L\qquad\text{(For } n=0\text{, it's } 2La_0.)$$

Use parametric differentiation to do this.

$$\int_{-L}^{L}\cos\alpha x=\frac{2}{\alpha}\sin\alpha L\qquad\text{take four derivatives:}$$

$$\frac{2\cdot 4!}{\alpha^5}\sin\alpha L-4\frac{2\cdot 3!}{\alpha^4}L\cos\alpha L-6\frac{2\cdot 2!}{\alpha^3}L^2\sin\alpha L+4\frac{2}{\alpha^2}L^3\cos\alpha L+\frac{2}{\alpha}L^4\sin\alpha L$$

$\alpha=n\pi/L$, and the sine terms are out. This is, for $n\neq 0$,

$$(-1)^nL^5\left[\frac{-48}{n^4\pi^4}+\frac{8}{n^2\pi^2}\right]=La_n$$

The $n=0$ case is simply $2L^5/5=2La_0$. Put this into the Fourier series to get

$$x^4=\frac{1}{5}L^4+L^4\sum_1^\infty(-1)^n\left[\frac{8}{n^2\pi^2}-\frac{48}{n^4\pi^4}\right]\cos n\pi x/L$$

That this behaves as $1/n^2$ for large n is a reflection of the fact that the derivative of the function being expanded is discontinuous at $x=L$. Evaluate this at $x=L$.

$$L^4=\frac{1}{5}L^4+L^4\sum_1^\infty\left[\frac{8}{n^2\pi^2}-\frac{48}{n^4\pi^4}\right]$$

The series $\sum 1/n^2 = \pi^2/6$ from a previous calculation. Solve for the value of the other series.

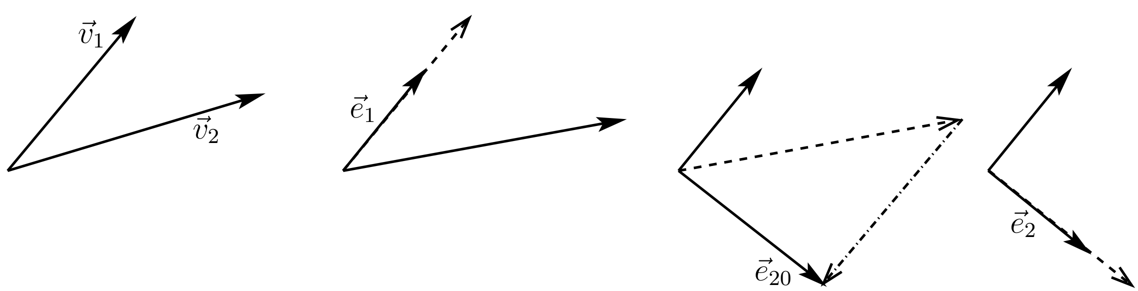
$$L^4 = \frac{1}{5}L^4 + \frac{4}{3}L^4 - L^4\sum_1^\infty \frac{48}{n^4\pi^4} \qquad \text{then} \qquad \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$$

5.37 The boundary conditions on $u'' = \lambda u$ are now $u(0) = 0$ and $2u(L) = Lu'(L)$. If $\lambda < 0$ the solutions are $\sin kx$, and $2\sin kL = kL\cos kL$. There are many such solutions. (Draw graphs!)
If $\lambda = 0$ the solution is $u = kx$ and $2kL = kL$. There is no such solution.
If $\lambda > 0$ the solution is $\sinh kx$ and $2\sinh kL = kL\cosh kL$. There is one value of k that this allows. You can find it by iteration on $kL = 2\tanh kL$. Draw a graph of the two sides of the equation, and you see that they cross in the neighborhood of $kL = 2$. Start the iteration there.

$$\begin{aligned} kL = 2 \to kL &= 2\coth 2 = 1.9281 \\ &\to\; 2\coth 1.9281 = 1.91715, \\ &\to\; 2\coth 1.91715 = 1.91536 \\ &\to\; 2\coth 1.91536 = 1.91507 \end{aligned}$$

A few more iterations (easy enough if you have $\tanh$ on a pocket or desktop calculator) gives 1.91500805. This is unusual in that you have eigenvalues of both signs in the same problem, leading to both circular sines and a hyperbolic sine. For equations more complicated than $u'' = \lambda u$, this phenomenon is more common, and in as simple an atom as hydrogen, the corresponding differential equation (Schroedinger's) has a infinite number of both positive and negative eigenvalues.

6.3



6.5 For the minimum of this function of $\lambda = x + iy$ arising during the proof of the Cauchy-Schwartz inequality, take its derivative with respect to x and y and set them to zero.

$$\begin{aligned} f(x,y) &= \langle \vec u - \lambda\vec v, \vec u - \lambda\vec v\rangle \\ &= \langle \vec u, \vec u\,\rangle + (x^2+y^2)\langle \vec v, \vec v\,\rangle - (x+iy)\langle \vec u, \vec v\,\rangle - (x-iy)\langle \vec v, \vec u\,\rangle, \\ \frac{\partial}{\partial x} &\to 2x\langle \vec v, \vec v\,\rangle - \langle \vec u, \vec v\,\rangle - \langle \vec v, \vec u\,\rangle = 0 \qquad \text{and} \\ \frac{\partial}{\partial y} &\to 2y\langle \vec v, \vec v\,\rangle - i\langle \vec u, \vec v\,\rangle + i\langle \vec v, \vec u\,\rangle = 0 \end{aligned}$$

$$x\langle \vec v, \vec v\,\rangle = \frac{1}{2}\big[\langle \vec u, \vec v\,\rangle + \langle \vec v, \vec u\,\rangle\big] = \Re\big(\langle \vec u, \vec v\,\rangle\big)$$

$$\text{and}\quad y\langle \vec v, \vec v\,\rangle = \frac{-i}{2}\big[\langle \vec v, \vec u\,\rangle - \langle \vec u, \vec v\,\rangle\big] = \Im\big(\langle \vec u, \vec v\,\rangle\big)$$

These are the real and imaginary parts of $\langle \vec u, \vec v\,\rangle$, so the combination is then $\lambda = x+iy = \langle \vec u, \vec v\rangle/\langle \vec v, \vec v\rangle$.

6.6 $\vec v_1 = \hat x + \hat y, \quad \vec v_2 = \hat y + \hat z, \quad \vec v_3 = \hat z + \hat x$ Use Gram-Schmidt:

$$\begin{aligned}
\vec e_1 &= \vec v_1/v_1 = \big(\hat x + \hat y\big)/\sqrt{2} \\
\vec e_{20} &= \vec v_2 - \vec e_1\big(\vec e_1 \cdot v_2\big) = \hat y + \hat z - \big[(\hat x + \hat y)/\sqrt{2}\big]\big[(\hat x + \hat y)/\sqrt{2}\big]\cdot\big[\hat y + \hat z\big] \\
&\qquad\qquad\qquad = \hat y + \hat z - \big[(\hat x + \hat y)/\sqrt{2}\big]/\sqrt{2} = -\tfrac{1}{2}\hat x + \tfrac{1}{2}\hat y + \hat z \\
\vec e_2 &= \big[-\tfrac{1}{2}\hat x + \tfrac{1}{2}\hat y + \hat z\big]\big/\sqrt{3/2} \\
\vec e_{30} &= \vec v_3 - \vec e_1\big(\vec e_1\cdot v_3\big) - \vec e_2\big(\vec e_2 \cdot v_3\big) \\
&= \hat z + \hat x - \big[(\hat x + \hat y)/\sqrt{2}\big]\big[(\hat x + \hat y)/\sqrt{2}\big]\cdot\big[\hat z + \hat x\big] \\
&\qquad\qquad - \sqrt{\tfrac{2}{3}}\big[-\tfrac{1}{2}\hat x + \tfrac{1}{2}\hat y + \hat z\big]\sqrt{\tfrac{2}{3}}\big[-\tfrac{1}{2}\hat x + \tfrac{1}{2}\hat y + \hat z\big]\cdot\big[\hat z + \hat x\big] \\
&= \hat z + \hat x - \big[(\hat x + \hat y)/\sqrt{2}\big]/\sqrt{2} - \sqrt{\tfrac{2}{3}}\big[-\tfrac{1}{2}\hat x + \tfrac{1}{2}\hat y + \hat z\big]\sqrt{\tfrac{2}{3}}\,\tfrac{1}{2} \\
&= \tfrac{2}{3}\hat x - \tfrac{2}{3}\hat y + \tfrac{2}{3}\hat z \\
\vec e_3 &= \tfrac{2}{3}\big[\hat x - \hat y + \hat z\big]\big/\sqrt{4/3} = \big[\hat x - \hat y + \hat z\big]/\sqrt{3}
\end{aligned}$$

After the computation is over, it's easy to check that the three $\vec e$'s are orthogonal and normalized.

6.11 (a) and (b) are different only if you say that a polynomial having degree 3 requires that the coefficient of the x^3 term isn't zero. Some people will make this distinction, but I think it causes more trouble than it's worth.
(c) is a vector space and (d) is not, because $f(2) = f(1) + 1$ does not imply $\alpha f(2) = \alpha f(1) + 1$.
(e) is and (f) is not because $(-1)f$ is not in the space.
(g) and (h) are different vector spaces because $\int_{-1}^{1} dx\, x\,(5x^3 - 3x) = 0$.

6.13 The parallelogram identity is

$$\|\vec u + \vec v\,\|^2 + \|\vec u - \vec v\,\|^2 = \langle \vec u + \vec v, \vec u + \vec v\,\rangle + \langle \vec u - \vec v, \vec u - \vec v\,\rangle = 2\langle \vec u, \vec u\,\rangle + 2\langle \vec v, \vec v\,\rangle + \text{terms that cancel}$$

and because the norm comes from the scalar product, that's the proof.

6.22 The functions $\sin^2 x$, $\cos^2 x$, and 1 are not linearly independent, so one of them must go. $\sin^2 x\cos^2 x = (1 - \cos^2 x)\cos^2 x$, so it is a combination of $\cos^2 x$ and $\cos^4 x$. A choice for basis is

$$\sin x,\quad \cos x,\quad \sin^2 x,\quad \cos^2 x,\quad \sin^4 x,\quad \cos^4 x$$

and that is six dimensions.

6.24 The functions are polynomials of degree ≤ 4 and satisfying $\int_{-1}^{1} dx\, x f(x) = 0$. Any even function of x satisfies the integral requirement, so 1, x^2, and x^4 are appropriate elements for a basis. Now look for a linear combination of x and x^3 that works too.

$$\int_{-1}^{1} dx\, x(\alpha x + \beta x^3) = \frac{2}{3}\alpha + \frac{2}{5}\beta = 0, \qquad \text{which implies} \qquad \beta = -\frac{5}{3}\alpha$$

The fourth element of the basis is then $x - {}^5\!/_3\, x^3$. The space is dimension four. Should you have anticipated the number four for the dimension? A moments thought to note that the polynomials without the constraint have dimension five. Next, they are orthogonal to one fixed function (x) and that drops the dimension.

6.25 Tenth degree polynomials form an 11-dimensional vector space. The triple root provides 3 constraints, so $11 - 3 = 8$ dimensions. It *is* a vector space because the triple root constraint is preserved under sums of polynomials and under multiplication by scalars.

6.27 Check axiom 7; that looks the most problematic.

$$7.\ (\alpha+\beta)\vec v = \alpha\vec v + \beta\vec v. \qquad f_3 = f_1 + f_2 \qquad \text{means} \qquad f_3(x) = Af_1(x-a) + Bf_2(x-b)$$

Let $\vec v = f$, then $\alpha\vec v = \alpha f$ and

$$\alpha\vec v + \beta\vec v = \alpha f + \beta f = f_3 \qquad \text{then} \qquad f_3(x) = A\alpha f(x-a) + B\beta f(x-b)$$

Is this equal to $(\alpha+\beta)f(x)$ for all x? Pick an f that's non-zero at only one point, say x_0, then for all x

$$(\alpha+\beta)f(x) = A\alpha f(x-a) + B\beta f(x-b)$$

Let $\beta = 0$ then this is true only if $a = 0$ and $A = 1$. Similarly $b = 0$ and $B = 1$, so this reduces to the standard case.
Do the same sort of manipulation for the definition $f_3(x) = f_1(x^3) + f_2(x^3)$. Again, let $f_1 = \alpha f$ and $f_2 = \beta f$, and

$$\alpha\vec v + \beta\vec v = \alpha f + \beta f = f_3 = (\alpha+\beta)f \qquad \text{and} \qquad f_3(x) = \alpha f(x^3) + \beta f(x^3) = (\alpha+\beta)f(x)$$

At $x = 0 \pm 1$ this works, but at any other value of x it requires f to be a constant.

6.30 The constant λ must be real and non-negative. (It could even be zero, reducing this to a familiar case.)

6.31 $\langle 1, x\rangle = \sqrt{\langle 1,1\rangle}\sqrt{\langle x,x\rangle}\cos\theta$. Now to evaluate all these products.

$$\langle 1, x\rangle = \int_0^1 x^2\,dx\,1\cdot x = \frac{1}{4}, \qquad \langle 1, 1\rangle = \int_0^1 x^2\,dx\,1^2 = \frac{1}{3}, \qquad \langle x, x\rangle = \int_0^1 x^2\,dx\,x^2 = \frac{1}{5}$$

Solve for $\cos\theta = ({}^1\!/_4)/\sqrt{({}^1\!/_3)({}^1\!/_5)} = \sqrt{15}\,/4$, so $\theta = 14.48°$.
For the earlier scalar product, $\langle 1, x\rangle$ is an odd function integrated from -1 to $+1$. The result is zero, so the angle in this case is $90°$.

7.3

$$a\cdot d + \frac{1}{2}b\cdot d + \frac{1}{2}a\cdot c - \frac{1}{2}a\cdot c - \frac{1}{2}b\cdot d - b\cdot c = a\cdot d - b\cdot c$$

7.8 $\vec{e}_0 = 1$, $\vec{e}_1 = x$, $\vec{e}_2 = x^2$, $\vec{e}_3 = x^3$.

$$\frac{d}{dx}\vec{e}_0 = 0, \quad \frac{d}{dx}\vec{e}_1 = 1 = \vec{e}_0, \quad \frac{d}{dx}\vec{e}_2 = 2x = 2\vec{e}_1, \quad \frac{d}{dx}\vec{e}_3 = 3x^2 = 3\vec{e}_2$$

These determine the respective columns of the matrix of components of d/dx.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{whose square is the components of } d^2/dx^2\text{:} \qquad \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

7.9 Use the Legendre polynomials for a basis, and

$$\frac{d}{dx}\vec{e}_0 = 0, \quad \frac{d}{dx}\vec{e}_1 = 1 = \vec{e}_0, \quad \frac{d}{dx}\vec{e}_2 = 3x = 3\vec{e}_1, \quad \frac{d}{dx}\vec{e}_3 = {}^{15}\!/_2 x^2 - {}^3\!/_2 = 5\vec{e}_2 + \vec{e}_0$$

The components of this operator is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{whose square is the components of } d^2/dx^2\text{:} \qquad \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

7.10 $\det\left(A^{-1}\right) = 1\big/\det\left(A\right)$. This is so because if A takes the unit square into a parallelogram, the inverse operator A^{-1} takes the parallelogram back to the square. The ratio of areas is inverted.

7.16 The basic definition of the inertia tensor is as the operator

$$I(\vec{\omega}\,) = \int dm\,\vec{r} \times \left(\vec{\omega} \times \vec{r}\,\right) = \int dm\,\left(r^2\vec{\omega} - \vec{r}(\vec{\omega} \cdot \vec{r}\,)\right)$$

Substitute this into the supposed identity.

$$\vec{\omega}_1 \cdot I(\vec{\omega}_2) = \vec{\omega}_1 \cdot \int dm\,\left(r^2\vec{\omega}_2 - \vec{r}(\vec{\omega}_2 \cdot \vec{r}\,)\right) = \int dm\,\left(r^2\vec{\omega}_1 \cdot \vec{\omega}_2 - \vec{\omega}_1 \cdot \vec{r}(\vec{\omega}_2 \cdot \vec{r}\,)\right)$$

This is clearly symmetric in the two ωs, so it is the same as $I(\vec{\omega}_1) \cdot \vec{\omega}_2$

7.27 For the basis of powers, $\vec{e}_k = x^k$ $(k = 0,\, 1,\, 2,\, 3)$, the translation operator gives

$$\begin{aligned} &T_a\vec{e}_0 = 1 = \vec{e}_0 && T_a\vec{e}_2 = x^2 - 2ax + a^2 = \vec{e}_2 - 2a\vec{e}_1 + a^2\vec{e}_0 \\ &T_a\vec{e}_1 = x - a = \vec{e}_1 - a\vec{e}_0 && T_a\vec{e}_3 = x^3 - 3ax^2 + 3a^2x - a^3 = \vec{e}_3 - 3a\vec{e}_2 + 3a^2\vec{e}_1 - a^3\vec{e}_0 \end{aligned}$$

These provide the columns of the matrix,

$$\begin{pmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -2a & 3a^2 \\ 0 & 0 & 1 & -3a \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{square it to get} \qquad \begin{pmatrix} 1 & -2a & 4a^2 & -8a^3 \\ 0 & 1 & -4a & 12a^2 \\ 0 & 0 & 1 & -6a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and this represents translation by $2a$. For the inverse, let $a \longrightarrow -a$, and the product of this matrix and the original is one.
If a is very large, then a function such as x^3 will translate into something that, near the origin, has a value near to $-a^3$. That dictates the resulting $\vec{e}_0$-component of the result. Similarly the function x^2 will have a value at the origin of a^2 after translation by a.

7.28

$$\begin{aligned} f(\hat{x}) &= \hat{x} \times \vec{B} = \hat{z}B_y - \hat{y}B_z \\ f(\hat{y}) &= \hat{y} \times \vec{B} = -\hat{z}B_x + \hat{x}B_z \\ f(\hat{z}) &= \hat{z} \times \vec{B} = \hat{y}B_x - \hat{x}B_y \end{aligned} \qquad \Longrightarrow \qquad (B) = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$$

For the eigenvectors, pick a basis so that $\hat{z}$ is along $\vec{B}$, then only the B_z element is present.

$$\begin{pmatrix} 0 & B_z & 0 \\ -B_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \lambda \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

The determinant of $(B - \lambda I)$ is $-\lambda^3 - \lambda B_z^2 = 0$ with roots $\lambda = 0, \pm iB_z$. The eigenvector for $\lambda = 0$ is $\hat{z}$. For the other two,

$$B_z v_y = \pm iB_z v_x, \qquad \text{and} \qquad -B_z v_x = \pm iB_z v_y$$

These are of course the same equation, with solution $v_y = \pm iv_x$. The eigenvectors are therefore $\hat{x} \pm i\hat{y}$.

7.31 The Cayley-Hamilton theorem in a (very) special case:

$$\begin{aligned} M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det\left(M - \lambda I\right) &= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + ad - bc \end{aligned}$$

Substitute M for λ.

$$\begin{aligned} &M^2 - M(a+d) + (ad-bc)I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+db \\ ac+dc & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

7.41 The eigenvalues and eigenvectors of two-dimensional rotations:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \lambda \begin{pmatrix} c \\ d \end{pmatrix} \qquad \text{requires}$$

$$\det\begin{pmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{pmatrix} = 0 = (\cos\alpha - \lambda)^2 + \sin^2\alpha = \lambda^2 - 2\lambda\cos\alpha + 1$$

The roots of this equation are $\lambda = \cos\alpha \pm \sqrt{\cos^2\alpha - 1} = \cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}$. The corresponding eigenvectors are $(\sin\alpha \neq 0)$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = e^{\pm i\alpha}\begin{pmatrix} c \\ d \end{pmatrix}, \quad \text{or} \quad c\cos\alpha - d\sin\alpha = e^{\pm i\alpha}c \quad \Rightarrow \quad -d = \pm ic$$

Write out the column matrices for the eigenvectors and then translate them into the common vector notation.

$$e^{i\alpha} \to \begin{pmatrix} 1 \\ -i \end{pmatrix} \to \hat{x} - i\hat{y}, \qquad e^{-i\alpha} \to \begin{pmatrix} 1 \\ i \end{pmatrix} \to \hat{x} + i\hat{y}$$

7.47 The cofactor method says to multiply the elements of a column by the determinant of the corresponding minor — itself a determinant of one lower rank. Each increase in the dimension then multiplies the number of multiplications by that dimension. In other words, $n!$ products for an $n \times n$ determinant.

$$\begin{aligned}
&10! \to 3.6\times 10^6 \times 10^{-10}\,\text{sec} = 10^{-4}\,\text{sec} \\
&20! \to 2.4\times 10^{18} \times 10^{-10}\,\text{sec} = 10^{8} = 1\,\text{year} \\
&30! \to 2.6\times 10^{32} \times 10^{-10}\,\text{sec} = 10^{22} = 10^{14}\,\text{year} = 10\,000 \times \text{age of universe}
\end{aligned}$$

Gauss elimination requires fewer multiplications. The number required is

$$n(n-1) + (n-1)(n-2) + \cdots < n^3$$

$$\begin{aligned}
&10^3 \times 10^{-10}\,\text{sec} = 10^{-7}\,\text{sec} \\
&20^3 \to 8\times 10^{-7}\,\text{sec} \\
&30^3 \to 27\times 10^{-7}\,\text{sec} \\
&100^3 \to 10^{-4}\,\text{sec} \\
&1000^3 \to 10^{-1}\,\text{sec}
\end{aligned}$$

The contrast is striking.

8.4 $x = u + v$, and $y = u - v$. Use Eq. (8.6) with $f \to y$; $x' \to x$; $(x, y) \to (x, y)$; First $y' \to u$ then second $y' \to v$

$$\begin{aligned}
\frac{\partial y}{\partial x}\bigg|_u &= \left(\frac{\partial y}{\partial x}\right)_y \left(\frac{\partial x}{\partial x}\right)_u + \left(\frac{\partial y}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_u = 0\cdot 1 + 1\cdot(-1) = -1 \\
\frac{\partial y}{\partial x}\bigg|_v &= \left(\frac{\partial y}{\partial x}\right)_y \left(\frac{\partial x}{\partial x}\right)_v + \left(\frac{\partial y}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_v = 0\cdot 1 + 1\cdot 1 = 1
\end{aligned}$$

As a verification of this calculation, do it without using the chain rule, first solving for y in terms of x and u: $x + y = 2u$. Now it's obvious that $\partial y/\partial x\big|_u = -1$.
Similarly $x - y = 2v$, giving the other equation.

8.8 For the two resistors in parallel, the power is P:

$$I = I_1 + I_1, \qquad \text{and} \qquad P = I_1^2 R_1 + I_2^2 R_2$$

Minimize this, eliminating I_2.

$$P = R_1 I_1^2 + R_2(I - I_1)^2, \qquad \text{then} \qquad \frac{dP}{dI_1} = 2R_1 I_1 + 2R_2(I_1 - I) = 0$$
$$\Rightarrow \quad I_1 = IR_2/(R_1 + R_2)$$

The original equations were symmetric under the interchange of indices $1 \leftrightarrow 2$, so the solutions are too: $I_2 = IR_1/(R_1 + R_2)$. Now it's easy to see that $I_1 R_1 = I_2 R_2$. The minimum power consumption occurs when the voltages in the parallel resistors match. *Is* this a minimum? The power, P, is a quadratic in I_1 with a positive coefficient on the squared term. That makes this a minimum.

8.10 The kinetic energy of the drumhead is, for $z = A\, r\big(1 - r^2/R^2\big)\sin\theta\cos\omega_2 t$

$$\begin{aligned}
\int dA \frac{1}{2}\sigma \dot{z}^2 &= \int dA \frac{1}{2}\sigma A^2 r^2 (1 - r^2/R^2)^2 \sin^2\theta\, \omega_2^2 \sin^2\omega_2 t \\
&= \frac{1}{2}\sigma A^2 \omega_2^2 \sin^2\omega_2 t \int_0^R r\,dr\, r^2 (1 - r^2/R^2)^2 \int_0^{2\pi} d\theta \sin^2\theta \\
&= \frac{\pi}{4}\sigma A^2 \omega_2^2 \sin^2\omega_2 t \int_0^{r=R} du\, u(1 - u/R^2)^2 \\
&= \frac{\pi}{4} A^2 \omega_2^2 \sin^2\omega_2 t \left[\frac{1}{2}u^2 - \frac{2}{3}u^3/R^2 + \frac{1}{4}u^4/R^4\right]_0^{r=R} \\
&= \frac{\pi}{4}\sigma A^2 \omega_2^2 \sin^2\omega_2 t \left[\frac{1}{2}R^4 - \frac{2}{3}R^4 + \frac{1}{4}R^4\right] = \frac{\pi}{48}\sigma A^2 \omega_2^2 R^4 \sin^2\omega_2 t
\end{aligned}$$

8.11 The potential energy for the mode $z = z_0\big(1 - r^2/R^2\big)\cos\omega t$ is

$$\begin{aligned}
\int dA\, \frac{1}{2}T\big(\nabla z\big)^2 &= \int dA\, \frac{1}{2}T\big(\hat{r} z_0 2r/R^2 \cos\omega t\big)^2 \\
&= \frac{1}{2}T z_0^2 \cos^2\omega t \int_0^R 2\pi r\,dr\, 4r^2/R^4 = \pi T z_0^2 \cos^2\omega t
\end{aligned}$$

The sum of the kinetic and potential energy is

$$\pi T z_0^2 \cos^2\omega t + \frac{1}{6}\sigma R^2 \pi z_0^2 \omega^2 \sin^2\omega t$$

For this to be constant, the coefficients of $\sin^2$ and $\cos^2$ must match.

$$\pi T z_0^2 = \frac{1}{6}\sigma R^2 \pi z_0^2 \omega^2 \qquad \text{or} \qquad \omega^2 = 6T/\sigma R^2$$

8.29 Minimize the heat generation in the three resistors in parallel. Use Lagrange multipliers.

$$P = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3, \qquad \text{and} \qquad I_1 + I_2 + I_3 = I$$
$$\text{Then} \qquad \frac{\partial}{\partial I_1}\big[I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 - \lambda(I - I_1 - I_2 - I_3)\big] = 0$$

with similar equations for derivatives with respect to I_2 and I_3. The four equations are then

$$2I_1R_1 + \lambda = 0, \qquad 2I_2R_2 + \lambda = 0, \qquad 2I_3R_3 + \lambda = 0, \qquad I_1 + I_2 + I_3 = I$$

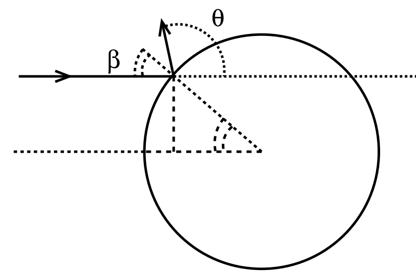
Without any further fuss, this tells you that $I_1R_1 = I_2R_2 = I_3R_3$. The parameter λ is, except for a factor $-1/2$, the common voltage across the resistors.

8.33 You can of course do the gradient in rectangular coordinates, but this is

$$\nabla r^2e^{-r} = \hat r\frac{\partial}{\partial r}r^2e^{-r} = \hat r\big[2r - r^2\big]e^{-r}$$

8.34 Use the same parametrization as the picture with Eq. (8.44),

$$b = R\sin\beta, \qquad \theta = \pi - 2\beta$$
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{R\sin\beta}{\sin(\pi-2\beta)}R\cos\beta\cdot\frac{1}{2} = \frac{R^2}{4}$$



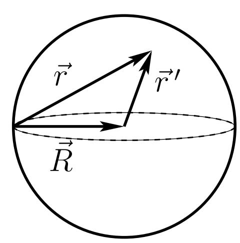
The total cross section is $\int d\Omega\, R^2/4 = \pi R^2$.

8.37 Assuming only one b for a given θ, and that $db/d\theta$ exists, then $db/d\theta$ will not change sign. In what follows then there can be an overall $\pm$ that will make everything positive.

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}, \qquad \text{so} \qquad \sigma = \int d\Omega\,\frac{b}{\sin\theta}\frac{db}{d\theta}$$
$$= \int \sin\theta\, d\theta\, d\phi\,\frac{b}{\sin\theta}\frac{db}{d\theta} = 2\pi\int d\theta\, b\frac{db}{d\theta} = 2\pi\int b\,db = \pi b^2_{\max}$$

8.49 The vector $\vec r$ from a point on the surface to one inside is $\vec r = \vec R + \vec r\,'$. Then, $r^2 = R^2 + r'^2 + 2\vec R\cdot\vec r\,'$. The volume integral is

$$\int r^2\,dV = \int dV\,\big(R^2 + r'^2 + 2Rr'\cos\theta\big)$$



The average value of the cosine over its range is zero, so the last term vanishes. The first two are now easy.

$$\int r^2\,dV = \frac{4\pi}{3}R^5 + \int_0^R 4\pi r'^2\,dr'\,r'^2 = \frac{4\pi}{3}R^5 + \frac{4\pi R^5}{5}$$

Divide this by the total volume $4\pi R^3/3$ to get $8R^2/5$.

8.51 The angular terms are both odd. $\cos\theta$ and $\sin^3\phi$ integrate to zero over the sphere so those terms contribute nothing.

$$\int \rho\, dV = \int_0^R 4\pi r^2\, dr\, \rho_0\big(1 + r^2/R^2\big) = 4\pi\rho_0\big[R^3/3 + R^3/5\big] = 32\pi\rho_0 R^3/15$$

Note that the coefficients 1/2 and 1/4 are small enough that the density never becomes negative.

Footnote, section 9.2 The parabola is $y = x^2$. A general straight line is $y = mx + b$. This line will almost always intersect the parabola in two points, and the unique exception occurs when it is tangent. Solve these two equations simultaneously and you get a quadratic equation, $x^2 - mx - b = 0$. For there to be only one root requires that the discriminant ($m^2 + 4b$) is zero, and the rest of the quadratic formula is then $x = m/2$, or $m = 2x$. That is the value of the slope at the coordinate x. To handle higher powers, I don't know such a direct way, but you *can* use a geometric argument to derive the product rule and then use it to handle the higher exponents. Similarly geometric arguments will get the chain rule and all the rest of the apparatus to differentiate elementary functions

9.1 The geometry is the same as the example following Eq. (9.3) in the text, so

$$\begin{aligned}\Delta\text{flow}_k = \vec v \cdot \Delta\vec A_k &= v_0 \frac{x_k y_k}{b^2}\hat x \cdot a\Delta\ell_k\big(\hat x\cos\phi - \hat y\sin\phi\big) \\ &= v_0\frac{x_k y_k}{b^2} a\Delta\ell_k\cos\phi = v_0\frac{\ell_k\sin\phi\,\ell_k\cos\phi}{b^2}a\Delta\ell_k\cos\phi\end{aligned}$$

Sum over the $\Delta\ell_k$ and take the limit to get an integral.

$$\begin{aligned}\int_0^{b/\cos\phi} d\ell\, v_0\frac{a}{b^2}\ell^2\sin\phi\cos^2\phi &= v_0\frac{a}{b^2}\frac{\ell^3}{3}\sin\phi\cos^2\phi\bigg|_0^{b/\cos\phi} \\ &= v_0\frac{a}{3b^2}\left(\frac{b}{\cos\phi}\right)^3\sin\phi\cos^2\phi = v_0\frac{ab}{3}\tan\phi\end{aligned}$$

If $\phi = 0$ there is no flow, because the velocity of the fluid is zero where $x = 0$. As $\phi \to \pi/2$ this approaches infinity. That's because the velocity gets bigger as y gets large.

9.8

$$\text{Area} = \int_0^{\theta_0}\sin\theta\, d\theta\int_0^{2\pi} d\phi R^2 = 2\pi R^2(1 - \cos\theta_0)$$

(b) For small θ_0, this is approximately $2\pi R^2[1 - (1 - \theta_0^2/2)] = \pi(R\theta_0)^2$. This is the area of the small disk of radius $R\theta_0$.
The largest θ_0 can get is 2π. Then the area is $4\pi R^2$.
For the integrals of $\vec v = \hat r v_0\cos\theta\sin^2\theta$,

$$\begin{aligned}\int \vec v \cdot d\vec A &= R^2\int_0^{\theta_0}\sin\theta\, d\theta\int_0^{2\pi} d\phi\,\hat r\cdot\hat r v_0\cos\theta\sin^2\phi \\ &= v_0\pi R^2\int -d(\cos\theta)\,\cos\theta = v_0\pi R^2(1 - \cos^2\theta_0)/2\end{aligned}$$

The integral of $\vec{v}\times d\vec{A}$ is zero because $\vec{v}$ is parallel to $\hat{n}=\hat{r}$.

9.21 The source charge is spherically symmetric, so the electric field will be too. The reason for this is that if $\vec{E}$ has a non-radial component at some point, then rotate the entire system by π about an axis through this point and the origin. The charge distribution won't change, but the sideways components of $\vec{E}$ will reverse. That can't happen. The field strength will not depend on angle for a similar reason: Rotate the system about any other axis through the origin and it take $\vec{E}$ to another point. It's still radial and the charge hasn't changed. That means that the field strength hasn't changed either. $\vec{E}=\hat{r}E_r(r)$.

$$\nabla\cdot\vec{E}=\frac{1}{r^2}\frac{d(r^2E_r)}{dr}=\rho(r)/\epsilon_0=\begin{cases}\rho_0/\epsilon_0 & (0<r<R)\\ 0 & (R<r)\end{cases}$$

Integrate this.

$$E_r(r)=\begin{cases}\rho_0 r/3\epsilon_0+C_1/r^2 & (r<R)\\ C_2/r^2 & (R<r)\end{cases}$$

If C_1 is non-zero, you will have a singularity from a point charge at the origin. Non is specified in the given charge density; $C_1=0$. The field is continuous at $r=R$, for otherwise you have an infinite dE_r/dr and so an infinite charge density there.

$$\rho_0R/3\epsilon_0=C_2/R^2\implies C_2=\rho_0R^3/3\epsilon_0=Q/4\pi\epsilon_0$$

(b) The total energy in this field is the integral of the energy density over all space.

$$\begin{aligned}\int d^3r\,\frac{\epsilon_0E^2}{2}&=\int_0^\infty 4\pi r^2\,dr\,\frac{\epsilon_0E^2}{2}\\ &=\int_0^R 2\pi\epsilon_0r^2\,dr\,\left(Qr/4\pi R^3\epsilon_0\right)^2+\int_R^\infty 2\pi\epsilon_0r^2\,dr\,\left(Q/4\pi\epsilon_0r^2\right)^2\\ &=2\pi\epsilon_0\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left[\frac{r^5}{5R^6}\bigg|_0^R+\frac{-1}{r}\bigg|_R^\infty\right]=2\pi\epsilon_0\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{6}{5R}=\frac{3}{5}\cdot\frac{Q^2}{4\pi\epsilon_0R}\end{aligned}$$

(c) Assign all the mass of the electron to this energy by $E_0=mc^2$.

$$mc^2=\frac{3}{5}\cdot\frac{Q^2}{4\pi\epsilon_0R}\qquad\text{or}\qquad R=\frac{3}{5}\cdot\frac{e^2}{4\pi\epsilon_0mc^2}$$

Here I changed the charge Q to the conventional symbol for the elementary charge. The value of this is 1.7×10^{-15} m. The last factor (not including the 3/5) is called the "classical electron radius" because of its appearance in an early attempt to model the structure of the electron.

9.26 The gravitational field of a spherical mass distribution is

$$g_r(r)=\begin{cases}-GM/r^2 & (R<r)\\ -GMr/R^3 & (r<R)\end{cases}$$

The energy density is $u=g^2/8\pi G$, and the additional gravitational field that this produces is, from problem 9.14

$$\frac{-4\pi G}{r^2}\int_0^r dr'\,r'^2\rho(r'),\qquad\text{where}\qquad\rho=u/c^2$$

For the interior of this spherical mass, this is

$$\frac{-4\pi G}{r^2}\int_0^r dr'\,r'^2\big[-GMr'/R^3\big]^2/8\pi Gc^2 = -\frac{G^2M^2}{2c^2r^2R^6}\int_0^r dr'\,r'^4 = -\frac{G^2M^2r^3}{10c^2R^6}$$

At the surface of the sphere the ratio of this correction to the original field is

$$\frac{G^2M^2R^3}{10c^2R^6}\div\frac{GM}{R^2} = \frac{GM}{10Rc^2} \qquad \text{Yes, this is dimensionless}$$

For the sun, assuming that it is a uniform sphere (it isn't), $R = 700,000$ km and $M = 2\times 10^{30}$ kg. This ratio is 2×10^{-7}.
For this ratio to equal one, doubling the field, you have $R = GM/10c^2$. For the sun this is 150 meters. The Schwarzchild radius that appears in the general theory of relativity is $2GM/c^2$.

9.27 The gravitational field is independent of θ and ϕ, so only the r-derivative in the divergence is present.

$$\nabla\cdot\vec g = \frac{1}{r^2}\frac{d(r^2g_r)}{dr} = -4\pi G\rho = -4\pi G\big(g_r^2/8\pi Gc^2\big) = -g_r^2/2c^2$$

To solve this equation, multiply by r^2 and let $f(r) = r^2g_r(r)$.

$$\frac{df}{dr} = -\frac{1}{2c^2r^2}f^2 \qquad \text{separate variables, and} \qquad \frac{df}{f^2} = -\frac{1}{2c^2}\frac{dr}{r^2}$$

$$\text{so} \qquad -\frac{1}{f} = \frac{1}{2c^2}\cdot\frac{1}{r} + K \qquad \text{then} \qquad f = \frac{-2c^2r}{1+2Kc^2r} \qquad \text{and} \qquad g_r(r) = \frac{-2c^2}{r+2Kc^2r^2}$$

The requirement that this behave as $-GM/r^2$ for large r determines the constant $K = 1/GM$.

$$g_r(r) = \frac{-2c^2}{r+2c^2r^2/GM} = \frac{-GM}{r^2+GMr/2c^2} = \frac{-GM}{r(r+R)} \qquad \text{where} \qquad R = GM/2c^2$$

This is *less* singular than Newton's solution for a point mass; it goes only as $1/r$ at the origin instead of as $1/r^2$. This happens because the source of the field is the field itself, and for a sphere of radius r, most of that field is outside the surface of the sphere. None of that part of the field will contribute to the field, making it weaker than expected as $r\to 0$.
(b) For the sun, $M = 1.997\times 10^{30}$ kg, and $R = 740$ m. The Schwarzchild radius that appears in the general theory of relativity is four times this.

9.28 The total energy in the gravitational field of the preceding problem is

$$\int u\,dV = \int_0^\infty 4\pi r^2\,dr\,\frac{1}{8\pi G}\left(\frac{-GM}{r(r+R)}\right)^2 = \int_0^\infty dr\,\frac{GM^2}{2}\frac{r^2}{r^2(r+R)^2}$$
$$= \frac{GM^2}{2}\int_0^\infty dr\,\frac{1}{(r+R)^2} = \frac{GM^2}{2}\frac{1}{R} = \frac{GM^2}{2}\frac{2c^2}{GM} = Mc^2$$

9.30

9.31 For a point mass at coordinates $(0,0,d)$, the potential is $-GM/|\vec{r}-\hat{z}d|$. This is

$$\frac{-GM}{\sqrt{\left(\vec{r}-\hat{z}d\right)^2}} = \frac{-GM}{\sqrt{r^2-2dz+d^2}} = \frac{-GM}{\sqrt{r^2-2dr\cos\theta+d^2}}$$

In order to expand this for small d, use the binomial expansion and rearrange the expression to conform to that.

$$\begin{aligned}\frac{-GM}{r}\sqrt{1-2(d/r)\cos\theta+(d^2/r^2)} = \frac{-GM}{r}\bigg[1& \\ +\left(-\frac{1}{2}\right)\left(-2(d/r)\cos\theta+(d^2/r^2)\right)+\left(\frac{1}{2}\frac{3}{2}\frac{1}{2!}\right)\left(-2(d/r)\cos\theta+(d^2/r^2)\right)^2& \\ +\left(-\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!}\right)\left(-2(d/r)\cos\theta+(d^2/r^2)\right)^3+\cdots\bigg]&\end{aligned}$$

In order to keep terms consistently to order d^3/r^3, you need only some parts of the terms that I've written out.

$$\begin{aligned}&\frac{-GM}{r}\bigg[1+\left(-\frac{1}{2}\right)\left(-2(d/r)\cos\theta+(d^2/r^2)\right) \\ &+\left(\frac{1}{2}\frac{3}{2}\frac{1}{2!}\right)\left(4(d^2/r^2)\cos^2\theta-4(d^3/r^3)\cos\theta\right)+\left(-\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!}\right)\left(-8(d^3/r^3)\cos^3\theta\right)\bigg]\end{aligned}$$

Now collect all the terms of like order in powers of d/r.

$$-\frac{GM}{r}-\frac{GMd}{r^2}\big[\cos\theta\big]-\frac{GMd^2}{r^2}\big[\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2}\big]-\frac{GMd^3}{r^3}\big[\tfrac{5}{2}\cos^3\theta-\tfrac{3}{2}\cos\theta\big]-\cdots$$

Look back at Eq. (4.61) to see that the angular dependence consists of Legendre polynomials of $\cos\theta$.

9.35

$$\delta_{ij}\epsilon_{ijk}=0,\quad \epsilon_{mjk}\epsilon_{njk}=2\delta_{mn},\quad \partial_i x_i=3,\quad \partial_i x_j=\delta_{ij},\quad \epsilon_{ijk}\epsilon_{ijk}=6,\quad \delta_{ij}v_j=v_i$$

You can do the first of these by writing it out, but there's a trick that shows up so often in these manipulations that it's worth mentioning. The indices i and j are dummies. They're summed over, so you can call them anything you want. I'll call i j and I'll call j i. That leaves the sum alone, and it is

$$\delta_{ij}\epsilon_{ijk}=\delta_{ji}\epsilon_{jik}$$

Interchanging the indices on δ leaves it alone, but interchanging them on ϵ changes the sign. This is equal to minus itself, so it's zero.
For the second, if $m\neq n$, then there are no terms in the sum that are non-zero. If they are equal, there are two terms, $1^2+(-1)^2$.
The rest are simpler.
The last identity $\epsilon_{ijk}\epsilon_{mnk}=\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}$ is just enumeration: i and j must be different for a non-zero result on the left, say $(i,j)=(2,3)$. Then the sum on k contains only the term $k=1$, and (m,n) must be either $(2,3)$ or $(3,2)$. The two cases give the terms on the right. All other cases are the same.

10.2 The $n = 0$ solution is really the solution with separation constant zero and that is of the form Eq. (10.12): $T(x,t) = Ax + B$. Apply the boundary conditions of the example involving Eq. (10.10). $T(0,t) = A = 0$, and $R(L,t) = AL + B = 0$. The result is $A = B = 0$.

10.16 With a solution assumed to be in the form $\sum r^n\big(a_n \cos n\theta + b_n \sin n\theta\big)$, take the $\theta = 0$ line to be as indicated, aimed toward the split between the cylinders. Apply the boundary condition

$$V(R,\theta) = \sum_0^\infty R^n\big(a_n \cos n\theta + b_n \sin n\theta\big) = \begin{cases} V_0 & (0 < \theta < \pi) \\ -V_0 & (\pi < \theta < 2\pi) \end{cases}$$

You can anticipate that the resulting potential will be an odd function of θ because the boundary condition is, but let that go and simply use Fourier series to evaluate the coefficients.

$$\int_0^{2\pi} d\theta\, \cos m\theta\, V(R,\theta) = 0 = a_m R^m \pi \quad \text{or } 2\pi \text{ if } m = 0$$

The first integral is zero because cosine is an even function and V is odd over the domain of integration.

$$\int_0^{2\pi} d\theta\, \sin m\theta\, V(R,\theta) = b_m R^m \pi = \begin{cases} 0 & (m \text{ even}) \\ 4V_0\pi/m & (m \text{ odd}) \end{cases}$$

Let $m = 2k + 1$, and

$$V(r,\theta) = \frac{4V_0}{\pi} \sum_{k=0}^\infty \frac{1}{2k+1} \left(\frac{r}{R}\right)^{2k+1} \sin(2k+1)\theta$$

On the central axis, I expect the E-field to have magnitude about $2V_0/2R$, maybe more, because $2R$ is the diameter of the cylinder and $2V_0$ is the potential difference. The electric field at $r = 0$ comes from the $k = 0$ term alone.

$$\frac{4V_0}{\pi R} r \sin\theta = \frac{4V_0}{\pi R} y$$

Higher order terms have vanishing derivatives there. $-d/dy$ of this shows a field strength of $4V_0/\pi R$. That's slightly larger than my estimate because these are not parallel plates, and the metal curves in closer to the axis.

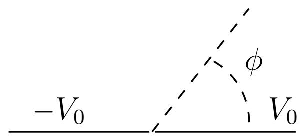
10.26 The solutions of Laplace's equation in cylindrical coordinates include the cases for which the separation constant is zero. In particular, $V(r,\theta) = A\theta + B$. For this problem, the potential is zero at $\theta = 0$ and V_0 at $\theta = \pi/2$. This is

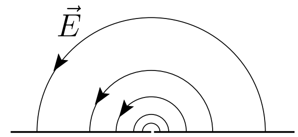
$$0 = A \cdot 0 + B, \qquad \text{and} \qquad V_0 = A\pi/2 + B$$

So $V(r,\theta) = 2V_0\theta/\pi$, and the electric field has $E_\theta = -(1/r)d/d\theta$ of V. That is, $\vec{E} = -2V_0\hat{\theta}/\pi r$

10.27 The solution of Laplace's equation in plane polar coordinates is Eq. (10.51). In particular $V = A_0 + B_0\phi$ will fit the boundary conditions of this problem. As drawn this is for above the plane, and at $\phi = 0$ the potential is $V_0 = A_0$. At $\phi = \pi$ the potential is $-V_0 = A_0 + B_0\pi$, so $B_0 = -2V_0/\pi$. The electric field is

$$\vec{E} = -\nabla V = -\hat{\phi}\frac{1}{r}\frac{dV}{d\phi} = \frac{2V_0}{\pi r}\hat{\phi}$$





The electric field below is the mirror image of the one above.

10.29

$$V(r,\theta) = \frac{4V_0}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{r}{R}\right)^{2k+1}\sin(2k+1)\theta$$

Temporarily drop the R, and let r replace r/R. I'll put it back at the end.
First sum the series $\sum_0^\infty \big[r^{2k+1}\sin(2k+1)\theta\big]/(2k+1)$. The imaginary part of

$$\sum_0^\infty \big[r^{2k+1}e^{i(2k+1)\theta}\big]/(2k+1) = \sum_0^\infty z^{2k+1}/(2k+1) = f(z)$$

Differentiate: $f'(z) = \sum_0^\infty z^{2k} = 1/(1-z^2)$. Now integrate, noting that $f(0) = 0$ and using Eq. (1.4)

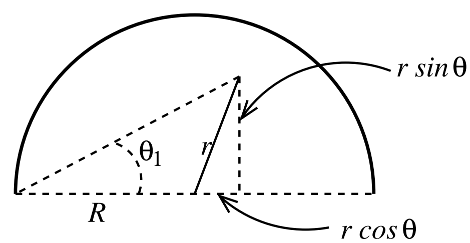
$$f(z) = \int_0^z \frac{dz}{1-z^2} = \tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z} = \frac{1}{2}\ln\frac{1+re^{i\theta}}{1-re^{i\theta}}$$
$$= \frac{1}{2}\ln\frac{1+r\cos\theta+ir\sin\theta}{1-r\cos\theta-ir\sin\theta} = \frac{1}{2}\left[\tan^{-1}\frac{r\sin\theta}{1+r\cos\theta} - \tan^{-1}\frac{-r\sin\theta}{1-r\cos\theta}\right]$$

The last equation is really the imaginary part of what preceded, because that's all that I want. Recall the logarithm, $\ln(re^{i\phi}) = \ln r + i\phi$. Reinstate the R factor in order to interpret this result

$$V(r,\theta) = \frac{2V_0}{\pi}\left[\tan^{-1}\frac{r\sin\theta}{R+r\cos\theta} + \tan^{-1}\frac{r\sin\theta}{R-r\cos\theta}\right]$$

Now draw a picture of R, r, and θ and interpret the numerators and the denominators. You immediately see that the arctangents are simply angles as measured from the two breaks in the boundary circle. The sketch shows θ_1, and θ_2 is at the other end of the diameter.

$$V(r,\theta) = 2V_0\big(\theta_1+\theta_2\big)/\pi$$



If you remember a theorem from Euclidean plane geometry, you can easily see that this matches the boundary conditions. The sum of the two angles θ_1 and θ_2, when the point is on the semicircle, is $90°$.

11.1 For a two point extrapolation formula, write the Taylor series expansions for the function. The data is given at $-h$ and at $-2h$.

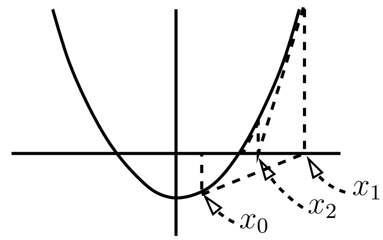
$f(-h) = f(0) - hf'(0) + (h^2/2)f''(0) -$ and $f(-2h) = f(0) - 2hf'(0) + (2h^2)f''(0) -$

$f(0)$ is the term you seek, so eliminate the largest term after that, the hf' term.

$$2f(-h) - f(-2h) = f(0) + (h^2 - 2h^2)f''(0) + \quad \text{so} \quad f(0) = 2f(-h) - f(-2h) + h^2 f''(0) + \cdots$$

11.3 Solve $f(x) = x^2 - a = 0$. Newton's method says $x_{k+1} = x_k - f(x_k)/f'(x_k) = x_k - (x_k^2 - a)/2x_k$. Start with a guess such as 0.5 and watch the sequence.

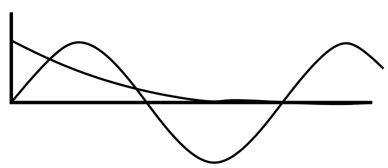
$$\begin{array}{lll} x_0 = .5 & x_1 = 2.35 & x_2 = 1.5694 \\ x_3 = 1.42189 & x_4 = 1.414234 & x_5 = 1.4142135625 \end{array}$$



as compared to $\sqrt{2} = 1.4142135623731$. A more intelligent initial choice will require fewer iterations, and a computer library routine that uses this method will optimize this choice.

11.4 Except for the first root, the roots of $e^{-x} = \sin x$ are near to $n\pi$ for positive integers n. Use Newton's method for these and return to the lowest root later.

$$\begin{aligned} f(x) &= e^{-x} - \sin x, \qquad \text{then for } x_0 = n\pi \\ x_1 &= n\pi - f(n\pi)/f'(n\pi) \\ &= n\pi - e^{-n\pi}/\big[-e^{-n\pi} - (-1)^n\big] \\ &= n\pi + 1/\big[(-1)^n e^{+n\pi} + 1\big] \end{aligned}$$



These roots are

$$n = 1: \quad \pi - 0.045166 = 3.096427, \qquad n = 2: \quad 2\pi + 0.001864, \qquad n = 3: \quad 3\pi - 0.0000807$$

For $n \geq 3$ the first correction to $n\pi$ is already below 10^{-4} so the higher order corrections will be far smaller. What about the first two?

$$x_1 = x_0 - \big[e^{-x_0} - \sin x_0\big]/\big[-e^{-x_0} - \cos x_0\big]$$

For $n = 1$ it is 3.0963639, a tiny change from the first order term, so I won't even bother with the corresponding correction to the next root.
For the single lowest root in the graph, it looks to be around $x_0 = 1$, so start there. The equation to iterate is the same.

$$x_1 = 1 - [\ ]/[\ ] = 0.4785, \qquad x_2 = 0.58415, \qquad x_3 = 0.588525, \qquad x_4 = 0.5885327$$

A more accurate sketch would probably have provided a more accurate starting point, but this converged anyway.

11.6 Use Simpson's rule to do the integral for $\operatorname{erf}(1)$. Take four points.

$$\frac{2}{\sqrt{\pi}}\frac{0.25}{3}\big[e^0 + 4e^{-1/16} + 2e^{-1/4} + 4e^{-9/16} + e^{-1}\big] = \frac{2}{\sqrt{\pi}}\frac{0.25}{3} \cdot 8.96226455749185 = 0.842736$$

A more accurate value for $\mathrm{erf}(1)$ is 0.842700792949715.

11.10 One or two point integration when the weighting function is e^{-x}. Assume the integration points are $x_{1,\,2,\,3}$ and that the weights for the points are α, β, γ.

$$\int_0^\infty dx\, e^{-x}\, f(x) = \alpha f(x_1) + \beta f(x_2) + \gamma f(x_3)$$

$$\begin{aligned}\int_0^\infty dx\, e^{-x}[f(0) + xf'(0) + x^2 f''(0)/2 + x^3 f'''(0)/6 + \cdots]&\\ = \alpha[f(0) + x_1 f'(0) + x_1^2 f''(0)/2 + x_1^2 f'''(0)/6 + \cdots]&\\ + \beta[f(0) + x_2 f'(0) + x_2^2 f''(0)/2 + x_2^2 f'''(0)/6 + \cdots]&\\ + \gamma[\cdots]&\end{aligned}$$

Do the integrals and match corresponding coefficients of f and as many of its derivatives as possible.

$$f(0) + f'(0) + f''(0) + f'''(0) + \cdots = \alpha[f(0) + x_1 f'(0) + x_1^2 f''(0)/2 + \cdots] + \beta\cdots$$

This is several equations.

$$\begin{aligned}1 &= \alpha + \beta + \gamma\\ 1 &= \alpha x_1 + \beta x_2 + \gamma x_3\\ 1 &= \tfrac{1}{2}\alpha x_1^2 + \tfrac{1}{2}\beta x_2^2 + \tfrac{1}{2}\gamma x_3^2\\ 1 &= \tfrac{1}{6}\alpha x_1^3 + \tfrac{1}{6}\beta x_2^3 + \cdots\end{aligned}$$

For the one point formula just set $\beta = \gamma = 0$ and you get $\alpha = 1$ and $x_1 = 1$.
For the two point formula set $\gamma = 0$ and you have four unknowns. Given the statement that the integration points are roots of $1 - 2x + \frac{1}{2}x^2 = 0$, you have $x_{1,\,2} = 2 \mp \sqrt{2} = 0.586,\ 3.414$.

$$\beta = 1 - \alpha, \qquad \text{then} \qquad 1 = \alpha\big(2 - \sqrt{2}\big) + (1 - \alpha)\big(2 + \sqrt{2}\big)$$

$$\alpha = \frac{2 + \sqrt{2}}{4} = 0.854, \qquad \beta = \frac{2 - \sqrt{2}}{4} = 0.146$$

11.11 $d\sin x/dx = \cos x$, and at $x = 1$ this is 0.5403023058681397. Compute it by a centered difference $[f(x + h) - f(x - h)]/2h$ where $x = 1$ and $h = 10^{-n}$ for $n = 1, 2, 3, \ldots$. The results are [approx − exact]

1 :	-8.5653592455298876E-02	1 :	-8.5653573E-02
2 :	-9.0005369837992122E-04	2 :	-8.9991093E-04
3 :	-9.0049934062391701E-06	3 :	-7.0333481E-06
4 :	-9.0050373838246323E-08	4 :	1.3828278E-05
5 :	-9.0056824497697363E-10	5 :	1.3828278E-05
6 :	-9.0591562029729289E-12	6 :	-8.8024139E-04
7 :	-3.8594127893532004E-14	7 :	-3.8604736E-03
8 :	1.3839193679920925E-11	8 :	0.3537674
9 :	-1.9432762343729593E-10		
10 :	-2.9698851850001873E-09		

This calculation was done using an accuracy of about 17 digits for the left set and about 8 for the second. You can see that the error is smallest at about $h = 10^{-7}$ in the first case and about $h = 10^{-3}$ for the second. Decreasing the interval beyond that point results in larger rather than smaller errors.

To analyze this analytically, assume that the function f has some fuzz attached to it: $f(x) \pm \epsilon$. In this example, ϵ is about 10^{-17} or 10^{-8} respectively. When you calculate the numerical derivative you are calculating

$$\frac{f(x+h) \pm \epsilon - f(x-h) \pm \epsilon}{2h}$$

The truncation error is stated in Eq. (11.8) to be $h^2 f'''(x)/6$. The error from roundoff is about ϵ/h, and as h decreases the truncation error goes down and the roundoff error goes up. The sum has a minimum when

$$\frac{d}{dh}\left[\frac{\epsilon}{h} - \frac{1}{6}h^2 f'''\right] = 0, \qquad \text{or} \qquad h = \left(\frac{3\epsilon}{|f'''(x)|}\right)^{1/3}$$

For the sine function with $\epsilon = 10^{-8}$ and $|f'''| = .5$ this is $h = 0.004$. For $\epsilon = 10^{-17}$ it is $h = 4 \times 10^{-6}$. These are in rough agreement with the numerical example above.

11.14 Try to minimize $F = \sum_i \left[y_i - \sum_\mu f_\mu(x_i)\right]^2$ subject to the constraint $G = \sum_\mu \alpha_\mu f_\mu(x_0) - K = 0$. This looks like a job for Lagrange multipliers. Minimize $F - \lambda G$; differentiate with respect to α_ν.

$$\frac{\partial}{\partial \alpha_\nu}(F - \lambda G) = -2\sum_i \left[y_i - \sum_\mu \alpha_\mu f_\mu(x_i)\right] f_\nu(x_i) - \lambda f_\nu(x_0) = 0$$

Save some factors of 2, and redefine $\lambda \to 2\lambda$. Rearrange the equations to be

$$\sum_\mu \left[\sum_i f_\nu(x_i) f_\mu(x_i)\right] \alpha_\mu = \sum_i y_i f_\nu(x_i) + \lambda f_\nu(x_0)$$

Following the notation of equations (11.50) and (11.51), this is

$$Ca = b + \lambda f_0, \qquad \text{where} \qquad f_0 \leftrightarrow f_\nu(x_0) \qquad \text{then} \qquad a = C^{-1}(b + \lambda f_0)$$

To solve for λ, take this solution for the column matrix, a and substitute it into the constraint $G = 0$.

$$G = \langle f_0, a\rangle - K = 0 = \left\langle f_0, \left[C^{-1}(b + \lambda f_0)\right]\right\rangle - K, \quad \text{then} \quad \lambda\langle f_0, C^{-1} f_0\rangle = K - \langle f_0, C^{-1} b\rangle$$

11.19

$$x = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)} \qquad \text{becomes} \qquad x = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

When you get close to the correct answer, both the numerator and the denominator are small. A little error in the numerator is magnified when you divide by a small number. In the first version, when x_2 is close to the right answer, the numerator in the second term is much smaller than the denominator so that the error magnification is less.

11.34 Subtract two square roots whose arguments are almost equal.

$$\sqrt{b+\epsilon} - \sqrt{b} = \sqrt{b}\left[\sqrt{1 + \epsilon/b} - 1\right] = \sqrt{b}\left[\epsilon/2b - \epsilon^2/8b^2\right] = \epsilon/2\sqrt{b} - \epsilon^2/8b^{3/2}$$

The second term at the end is the truncation error if you keep only the first term.

12.4 $\vec{d} = f(\vec{F})$, so if you choose the basis along the directions of the springs, the calculation becomes straight forward. $\hat{x}$ along one of the k_1 springs, and $\hat{y}$ along k_2 etc. Assume that the springs obey the usual $F_x = -kx$ relation, then a force along $\hat{x}$ gives a displacement $+F_x\hat{x}/2k_1$ because there are two springs in that direction.

$$f(\hat{x}) = \hat{x}/2k_1, \qquad f(\hat{y}) = \hat{y}/2k_2, \qquad f(\hat{z}) = \hat{z}/2k_3$$

and these show the components of f to be the diagonal matrix $\begin{pmatrix} 1/2k_1 & 0 & 0 \\ 0 & 1/2k_2 & 0 \\ 0 & 0 & 1/2k_3 \end{pmatrix}$

12.5 The components of a tensor are defined by $F(\vec{e}_i) = F_{ji}\vec{e}_j$. Let $F = ST$, then

$$F(\vec{e}_i) = S\big(T(\vec{e}_i)\big) = S\big(T_{ki}\vec{e}_k\big) = T_{ki}S(\vec{e}_k) = T_{ki}S_{jk}\vec{e}_j = F_{ji}\vec{e}_j$$

Equate the coefficients of the basis vectors on the two sides of the last equation to get

$$F_{ji} = S_{jk}T_{ki}$$

and this is matrix multiplication.

12.12 $\vec{e}_1 = 2\hat{x}$ and $\vec{e}_2 = \hat{x} + 2\hat{y}$.
The vector $\vec{e}^{\,2}$ is orthogonal to $\vec{e}_1$ so it is along $\hat{y}$. To make the scalar product with $\vec{e}_1$ equal to one, it must be $\vec{e}^{\,2} = \hat{y}/2$.
The vector $\vec{e}^{\,1}$ is orthogonal to $\vec{e}_2$ so it is along $2\hat{x} - \hat{y}$. To make the scalar product with $\vec{e}_1$ equal to one, make it $\vec{e}^{\,1} = (2\hat{x} - \hat{y})/4$.
The various dot products that you can take here are

$$\begin{gathered} \vec{e}_1 = 2\hat{x} \qquad \vec{e}_2 = \hat{x} + 2\hat{y} \qquad \vec{e}^{\,1} = (2\hat{x} - \hat{y})/4 \qquad \vec{e}^{\,2} = \hat{y}/2 \\ \vec{e}^{\,1}\cdot\vec{e}^{\,1} = 5/16, \qquad \vec{e}^{\,2}\cdot\vec{e}^{\,2} = 1/4, \qquad \vec{e}^{\,1}\cdot\vec{e}^{\,2} = \vec{e}^{\,2}\cdot\vec{e}^{\,1} = -1/8 \\ \vec{e}_1\cdot\vec{e}_1 = 4, \qquad \vec{e}_2\cdot\vec{e}_2 = 5, \qquad \vec{e}_1\cdot\vec{e}_2 = \vec{e}_2\cdot\vec{e}_1 = 2 \end{gathered}$$

$$\begin{aligned} \vec{A} &= \hat{x} - \hat{y} = -\tfrac{1}{2}\vec{e}_2 + \tfrac{3}{4}\vec{e}_1 \qquad & \vec{B} &= \hat{y} = \tfrac{1}{2}\vec{e}_2 - \tfrac{1}{4}\vec{e}_1 \\ \vec{A} &= \hat{x} - \hat{y} = 2\vec{e}^{\,1} + -\vec{e}^{\,2} \qquad & \vec{B} &= \hat{y} = 2\vec{e}^{\,2} \end{aligned}$$

$$\begin{aligned} \vec{A}\cdot\vec{B} &= \big(-\tfrac{1}{2}\vec{e}_2 + \tfrac{3}{4}\vec{e}_1\big)\cdot\big(\tfrac{1}{2}\vec{e}_2 - \tfrac{1}{4}\vec{e}_1\big) = -\tfrac{1}{4}\cdot 5 - \tfrac{3}{16}\cdot 4 + \tfrac{1}{8}\cdot 2 + \tfrac{3}{8}\cdot 2 = \tfrac{-16}{16} = -1 \\ \vec{A}\cdot\vec{B} &= \big(2\vec{e}^{\,1} - \vec{e}^{\,2}\big)\cdot\big(2\vec{e}^{\,2}\big) = 4\cdot -\tfrac{1}{8} - 2\cdot\tfrac{1}{4} = -1 \\ \vec{A}\cdot\vec{B} &= \big(-\tfrac{1}{2}\vec{e}_2 + \tfrac{3}{4}\vec{e}_1\big)\cdot\big(2\vec{e}^{\,2}\big) = -1 + 0 = -1 \\ \vec{A}\cdot\vec{B} &= \big(2\vec{e}^{\,1} - \vec{e}^{\,2}\big)\cdot\big(\tfrac{1}{2}\vec{e}_2 - \tfrac{1}{4}\vec{e}_1\big) = 2\cdot -\tfrac{1}{4} - 1\cdot\tfrac{1}{2} = -1 \end{aligned}$$

The scalar product is designed to be easiest in the last two cases, between mixed types of components.

12.15 If $T(\vec{v}, \vec{v}) = 0$ for all $\vec{v}$, then

$$T(\alpha\vec{u} + \beta\vec{v}, \alpha\vec{u} + \beta\vec{v}) = 0 = \alpha^2 T(\vec{u}, \vec{u}) + \alpha\beta\big[T(\vec{u}, \vec{v}) + T(\vec{v}, \vec{u})\big] + \beta^2 T(\vec{v}, \vec{v})$$

The first and last terms are zero, implying that the middle terms must add to zero: $T(\vec{u}, \vec{v}) + T(\vec{v}, \vec{u}) = 0$ and that is what was to be proved.

13.2 The width of the base of the parabola is $2a$. Its height is a^2/b. I can estimate the length by the triangle of this size: $2\sqrt{a^2+(a^2/b)^2} = 2a\sqrt{1+a^2/b^2}$. That's a lower bound. For an upper bound, the rectangle enclosing it has length $2a + 2a^2/b = 2a\big(1+a/b\big)$.
To compute the length of arc,

$$\int d\ell = \int_{-a}^{a} dx\,\sqrt{1+4x^2/b^2}$$

Let $2x/b = \sinh\theta$, then the integral is

$$\begin{aligned}\int \frac{b}{2}\cosh\theta\,d\theta\,\sqrt{1+\sinh^2\theta} &= \frac{b}{2}\int \cosh^2\theta\,d\theta = \frac{b}{2}\int d\theta\,(1+\cosh 2\theta)/2 \\ &= \frac{b}{4}\Big[\theta + {}^1\!/\!_2\sinh 2\theta\Big]_{x=-a}^{x=a} = \frac{b}{2}\Big[\theta + \sinh\theta\cosh\theta\Big]_{x=a} \\ &= \frac{b}{2}\big[\sinh^{-1}(2a/b) + (2a/b)\sqrt{1+(2a/b)^2}\big]\end{aligned}$$

If $a \ll b$ this is approximately $(b/2)\big[(2a/b)+(2a/b)\big] = 2a$. That agrees with both the upper and lower estimates that I started with.
If $b \ll a$ the inverse hyperbolic sine is small because is increases logarithmically. The other term is algebraic, so the result is approximately $(b/2)(2a/b)^2 = 2a^2/b$, again agreeing with both estimates.

13.3 To show that this is an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi + \sin^2\phi = 1$$

is a standard form for an ellipse.
To compute its area, make the change of variables $y' = ya/b$, then the element of area $dx\,dy = dx\,dy'\,a/b$ because the rectangular element of area is stretched by this factor. In these coordinates the equation of the curve is $x^2 + y'^2 = a^2$, and that's a circle of area πa^2. The original area is scaled by the factor b/a, so it is πab. The ellipse really *is* a squashed circle.
For the circumference,

$$\oint d\ell = \oint \sqrt{dx^2+dy^2} = \oint \sqrt{(a\sin\phi)^2 + (b\cos\phi)^2}\,d\phi = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,d\phi$$

Manipulate this now. Let $m = 1 - b^2/a^2$

$$= 4\int_0^{\pi/2} \sqrt{a^2 + (b^2-a^2)\cos^2\phi}\,d\phi = 4\int_0^{\pi/2} a\sqrt{1-(1-b^2/a^2)\cos^2\phi}\,d\phi$$

This is the complete elliptic integral of the second kind, Equation 17.3.3 in Abramowitz and Stegun. It doesn't matter whether it's a sine or a cosine in the integrand.

$$\text{Area} = 4aE(m)$$

Eq. 17.3.12 of A&S says that $E(0) = \pi/2$, so if $b = a$, this reduces to the circumference of a circle, $2\pi a$.
If $b \to 0$ then $m \to 1$, and $E(1) = 1$. The circumference becomes $4a$.

13.13 $\vec{F} = \hat{r}f(r,\theta,\phi) + \hat{\theta}g(r,\theta,\phi) + \hat{\phi}h(r,\theta,\phi)$ The line integral in this example is solely in the $\hat{\phi}$ direction, so $\vec{F}\cdot d\vec{\ell}$ contains only the term in h. The other side of Stokes' theorem involves the curl, and for that use Eq. (9.33).

$$\nabla\times\vec{F} = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, h)}{\partial\theta} - \frac{\partial g}{\partial\phi}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi} - \frac{1}{r}\frac{\partial(rh)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rg)}{\partial r} - \frac{\partial f}{\partial\theta}\right)$$

The surface integral has its normal vector along $\hat{r}$, so it is the integral

$$\int dA\,\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\, h)}{\partial\theta} - \frac{\partial g}{\partial\phi}\right)$$

Look at the second term, the one with g in it.

$$\int_0^{\pi}\frac{r^2\sin\theta}{r\sin\theta}d\theta\int_0^{2\pi}d\phi\,\frac{\partial g}{\partial\phi}\qquad\text{and the phi integral is}\qquad\int_0^{2\pi}d\phi\,\frac{\partial g}{\partial\phi} = g(r,\theta,\phi)\Big|_0^{2\pi}$$

If g is a function, that is, if it is single-valued, it has the same value at these two limits. That term vanishes, and the integral depends on h alone.

13.25 Translate this to index notation and it is

$$\begin{aligned}\big(\nabla\cdot(\vec{A}\times\vec{B})\big)_i = \partial_i\epsilon_{ijk}A_jB_k &= \epsilon_{ijk}(\partial_iA_j)B_k + \epsilon_{ijk}(\partial_iB_k)A_j\\ &= \epsilon_{kij}(\partial_iA_j)B_k - \epsilon_{jik}(\partial_iB_k)A_j\\ &= \big((\nabla\times\vec{A}\,)\cdot\vec{B} - (\nabla\times\vec{B}\,)\cdot\vec{A}\big)_i\end{aligned}$$

Here I used that fact that ϵ is unchanged under cyclic permutations of the indices and that it changes sign under interchange of any two.
Apply Gauss's theorem to this, changing the vector $\vec{A}$ to $\vec{v}$ to avoid confusion with area, then using the assumption that $\vec{B}$ is a constant vector to take it outside the integral.

$$\begin{aligned}\oint(\vec{v}\times\vec{B})\cdot d\vec{A} &= \int\nabla\cdot(\vec{v}\times\vec{B})dV\\ &= \int dV(\nabla\times\vec{v}\,)\cdot\vec{B} - (\nabla\times\vec{B}\,)\cdot\vec{A}\\ \oint(d\vec{A}\times\vec{v}\cdot\vec{B}) &= \vec{B}\cdot\int dV(\nabla\times\vec{v}\,)\end{aligned}$$

This used the fact that a cyclic permutation of the triple product leaves it unchanged. Now because $\vec{B}$ is an arbitrary vector, the factors times $\vec{B}\,\cdot$ must be equal.

$$\oint d\vec{A}\times\vec{v} = \int dV(\nabla\times\vec{v}\,)$$

13.26 $\partial_i(fF_i) = (\partial_if)F_i + f\partial_iF_i$ is simply the product rule for ordinary functions. Translate it into vector notation and it is $\nabla\cdot(f\vec{F}\,) = \vec{F}\cdot\nabla f + f\nabla\cdot\vec{F}$.
Integrate this over a volume and apply Gauss's theorem.

$$\int\nabla\cdot(f\vec{F}\,)dV = \oint f\vec{F}\cdot d\vec{A} = \int dV\,\big(\vec{F}\cdot\nabla f + f\nabla\cdot\vec{F}\,\big)$$

If $\vec{F}$ is a constant, I can pull it outside the integral.

$$\vec{F}\cdot\oint f d\vec{A} = \vec{F}\cdot\int dV\,\nabla f$$

This holds for all $\vec{F}$, so gives a result matching problem 13.6.

$$\oint f d\vec{A} = \int dV\,\nabla f$$

13.27 $f = -\gamma xyz^3/3$ works.

13.28 Find a vector potential for the given $\vec{B}$. I will choose $A_z = 0$.

$$\nabla\times\big(\hat{x}A_x + \hat{y}A_y\big) = -\hat{x}\partial_z A_y + \hat{y}\partial_z A_x + \hat{z}\big(\partial_x A_y - \partial_y A_x\big) = \alpha\hat{x}\,xy + \beta\hat{y}\,xy + \gamma\hat{z}\,(xz+yz)$$

$$-\partial_z A_y = \alpha xy, \qquad \partial_z A_x = \beta xy, \qquad \partial_x A_y - \partial_y A_x = \gamma(xz+yz)$$

It looks like something along the lines of xyz will work for both components, but I have to adjust the factors.
If $A_y = -\alpha xyz$ and $A_x = \beta xyz$, then the first two equations are satisfied. Now for the third.

$$\partial_x A_y - \partial_y A_x = -\alpha yz - \beta xz = \gamma(xz+yz)$$

This requires $\gamma = -\alpha = -\beta$. Then the vector potential is

$$\vec{A} = \gamma\big(-\hat{x}+\hat{y}\big)xyz \qquad \text{and} \qquad \vec{B} = \gamma\big(-\hat{x}\,xy - \hat{y}\,xy + \hat{z}\,(xz+yz)\big)$$

The divergence of the given $\vec{B}$ is zero if and only if $\gamma = -\alpha = -\beta$, precisely the same condition that I needed in order to find a vector potential.

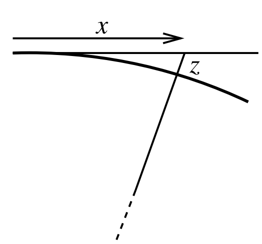
13.34 The air mass taken straight up is $\int_0^\infty dz\rho_0 e^{-z/h} = \rho_0 h$.
Looking toward the setting sun and ignoring refraction, this is $\int_0^\infty dx\rho_0 e^{-z/h}$, where x is measured starting horizontally, but in a straight line.

$$(R+z)^2 = x^2 + R^2, \qquad \text{so the air mass is} \qquad \int_0^\infty dx\,\rho_0 e^{-\left(\sqrt{R^2+x^2}-R\right)/h}$$

OR, expand the square root, $x \ll R$, and $\qquad z = x^2/2R$

the air mass is then

$$\int_0^\infty dx\,\rho_0 e^{-x^2/2Rh} = \rho_0\sqrt{2Rh\pi}\,/2$$



The ratio of the air mass toward the horizon and straight up is then $\sqrt{R\pi/2h} = 32$. If you include refraction by the air, that will bend the light so that it passes through an even larger air mass.

(c) To get a worst-case estimate of how much refraction affects this result, assume that all the refraction takes place *at* the surface. The angle of refraction is $\theta \approx 0.5^\circ$, so the distance moved along the surface is $R\theta$, and the corresponding air mass is $\rho_0 R\theta$. Add this to the preceding result for the total and divide this by $\rho_0 h$ to compare it to the air mass straight up.

$$\rho_0 R\theta/\rho_0 h = R\theta/h = 6400 \times \theta/10 = 5.6$$

The total is then about 37, so the true answer is somewhere between these bounds 32 and 37.
If you want to do this by completely evaluating the original integral, the one with the exponential of $\sqrt{R^2 + x^2}$ in it, make the substitution $x^2 + R^2 = u^2$ and you will find

$$\rho_0 e^{R/h} \int_R^\infty \frac{u\,du}{\sqrt{u^2 - R^2}} e^{-u/h} = \rho_0 e^{R/h} R K_1(R/h)$$

where this comes from having a big enough table of integrals, such as Gradshteyn and Ryzhik, and its equation 3.365.2 gives the result as a form of Bessel function (K_1). That in turn you can evaluate with another equation from the same book, 8.451.6, and the first term of the resulting series is precisely the previous result $\rho_0\sqrt{2Rh\pi}\,/2$. The one thing you get from this more complicated solution is an estimate of the error. The next term in the series is a factor of h/R smaller than the first one.

13.35 Set the limits on the variables to V_1, V_2 and P_1, P_2. The work integral is then

$$W = \oint P\,dV = \int_{V_1}^{V_2} dV\,\frac{nRT}{V} + \int_{V_2}^{V_1} dV\,P_1 = nRT \ln\frac{V_2}{V_1} - P_1(V_2 - V_1)$$

where $T = P_1 V_2 = P_2 V_1$. When the pressure change and the volume change are small, the graph looks like a triangle, so the integral (which is the area enclosed) should be approximately $(P_2 - P_1)(V_2 - V_1)/2$. Is it? Let $\Delta P = P_2 - P_1$ and $\Delta V = V_2 - V_1$ and do power series expansions. It's easy to make a plausible assumption here and then to get the wrong answer. (I did.) The log is $\ln(1 + x) \approx x$.

$$\begin{aligned} W = nRT\ln\left(1 + \frac{\Delta V}{V_1}\right) - P_1\Delta V &\approx nRT\frac{\Delta V}{V_1} - P_1\Delta V \\ &= P_2 V_1\frac{\Delta V}{V_1} - P_1\Delta V = (P_2 - P_1)\Delta V = \Delta P\Delta V \end{aligned}$$

This disagrees with what I expected. The area of a triangle has a factor $1/2$ in it. What went wrong? Answer: $\ln(1 + x) = x - x^2/2 + \cdots$ and the x^2 term matters.

$$\begin{aligned} W \approx nRT\left[\frac{\Delta V}{V_1} - \frac{1}{2}\left(\frac{\Delta V}{V_1}\right)^2\right] - P_1\Delta V &= P_2 V_1\frac{\Delta V}{V_1} - \frac{1}{2}P_2 V_1\frac{\Delta V^2}{V_1^2} - P_1\Delta V \\ &= \Delta P\Delta V - \frac{1}{2}P_2\frac{\Delta V^2}{V_1} \end{aligned}$$

The last term is the same order (second) as the ones that I kept before. I can't ignore the second order term in the expansion of the logarithm. Now to manipulate that final term: Along the isothermal line, $PV =$constant, so $P\,dV + V\,dP = 0$ to first order, but watch the signs! This is $P(V_2 - V_1) = +V(P_2 - P_1)$ because the dP and dV refer to the changes in the variables along the curve. When one goes down the other goes up. To this order it doesn't matter whether I use P_1 or P_2 as a factor; the effect on the result would be in the third order.

$$W \approx \Delta P\Delta V - \frac{1}{2}P_2\frac{\Delta V^2}{V_1} = \Delta P\Delta V - \frac{1}{2}V_1\Delta P\frac{\Delta V^2}{V_1} = \frac{1}{2}\Delta P\Delta V$$

13.37 The field $\vec{F}$ is a curl, so its divergence is zero. Now apply the divergence theorem.

$$\int_{\text{hemisphere}} + \int_{\text{bottom disk}} = \int d^3r\,\nabla\cdot\vec{F} = 0$$

solve for the desired integral to get

$$\int_{\text{hemisphere}} = -\int_{\text{disk}} = +\int dA\,\big(\nabla\times(\alpha y\hat{x} + \beta x\hat{y} + \gamma xy\hat{z})\big)_z = +\int dA\,(\beta-\alpha) = (\beta-\alpha)\pi R^2$$

13.39 $F(r,\theta) = r^n\big(A + B\cos\theta + C\cos^2\theta\big)$. Ref: Eq. (9.16).

$$\begin{aligned}\nabla F &= \hat{r}\frac{\partial F}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial F}{\partial\theta}\\ &= \hat{r}nr^{n-1}\big(A + B\cos\theta + C\cos^2\theta\big) - \hat{\theta}r^{n-1}\big(B\sin\theta + 2C\cos\theta\sin\theta\big)\\ \nabla\cdot(\text{this}) &= \frac{1}{r^2}\frac{\partial r^2(\text{this})_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta(\text{this})_\theta)}{\partial\theta}\\ &= n(n+1)r^{n-2}\big(A + B\cos\theta + C\cos^2\theta\big) + r^{n-2}(-2B\cos\theta + 2C\sin^2\theta - 4C\cos^2\theta)\\ &= r^{n-2}\big[An(n+1) + B(n^2+n-2)\cos\theta + 2C + C(n^2+n-6)\cos^2\theta\big]\end{aligned}$$

For this to be zero, then if B and $C = 0$ then $n = 0, -1$, giving solutions proportional to 1 or $1/r$.
If A and $C = 0$ then $n^2 + n - 2 = (n+2)(n-1) = 0$, giving solutions proportional to $r\cos\theta$ or $r^{-2}\cos\theta$.
If $C \neq 0$ then $n(n+1) - 6 = (n-2)(n+3) = 0$. Also $B = 0$ and $An(n+1) + 2C = 0$. The last equation is also $6A + 2C = 0$. This determines $A = -C/3$, and if you now choose $C = 3/2$ you get solutions proportional to $r^2\big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\big)$ and $r^{-3}\big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\big)$.

13.41 One way is to use the divergence theorem to evaluate $\int\vec{F}\cdot d\vec{A}$ over the hemisphere.

$$\oint\vec{F}\cdot d\vec{A} = \int_{\text{hemisphere}} + \int_{\text{disk}} = \int d^3r\,\nabla\cdot\vec{F} = \int d^3r\,(A+B+C) = (A+B+C)2\pi/3$$

Solve for the integral over the hemisphere to get

$$\begin{aligned}\int_{\text{hemisphere}} &= (A+B+C)2\pi/3 - \int dA(-1)C(1+x)\\ &= (A+B+C)2\pi/3 + C\pi = (2A+2B+5C)\pi/3\end{aligned}$$

14.3 For $x \neq 0$ the derivative of e^{-1/x^2} involves x^{-3} and the same exponential. Any higher derivative will also be in the form of some inverse powers of x times the original e^{-1/x^2}. What happens as $x \to 0$ for such a product?

$$\lim_{x\to 0} e^{-1/x^2}/x^n = \lim_{y\to\infty} y^{n/2}\,e^{-y} = 0$$

The exponential always wins. I could leave it here, but there's a subtle point that I should address. The above calculation shows that the limit of the n^{th} derivative as $x \to 0$ is zero. Does that prove that the derivative *at* zero is zero? With common functions you're used to assuming that derivatives are continuous, but this function shows some pathologies, so I have to ask if this is right. (It is, but sometimes you have to check.) To take the derivative of a function $g(x)$ at zero, you take the limit of $\big[g(x) - g(0)\big]/x$. If $g(0) = 0$, this is just $\lim g(x)/x$ and if $g(x)$ is of the form e^{-1/x^2} times any positive or negative powers of x the limit is zero.
Use induction. $f(0) = 0$, so let $g(x) = f(x)$ and $f'(0) = 0$. Now assume that the n^{th} derivative at the origin vanishes, $f^{(n)}(0) = 0$, then let $g(x) = f^{(n)}(x)$ and so $f^{(n+1)}(0) = 0$. This is a fine point, but it does make for a complete proof.
The Taylor series about zero has every coefficient equal to zero, so of course the series converges and in fact converges for all values of x. It just doesn't converge to the function you expect.

14.5

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z-i+2i)} = \frac{1}{(z-i)(2i)\big(1+(z-i)/2i\big)} = \frac{1}{z-i}\,\frac{-i}{2}\sum_0^\infty (-1)^k \left(\frac{z-i}{2i}\right)^k$$

This converges if $\big|(z-i)/2i\big| < 1$. That is a disk of radius 2 centered at i. Another series expansion about this point is

$$\frac{1}{(z-i)^2\big(1+2i/(z-i)\big)} = \frac{1}{(z-i)^2}\sum_0^\infty (-1)^k \left(\frac{2i}{z-i}\right)^k$$

This converges if $\big|2i/(z-i)\big| < 1$. This is the region *outside* the disk of radius 2 centered at $z = i$.

14.6 Use the substitution $z = \tanh\theta$.

$$\int_0^i dz\,\frac{1}{1-z^2} = \int_{z=0}^{z=i} \frac{\operatorname{sech}^2\theta}{1-\tanh^2\theta}\,d\theta = \theta\Big|_{z=0}^{z=i} = \tanh^{-1} i - 0 = i\tan^{-1} 1 = i\pi/2$$

14.7

$$\frac{1}{z^4}\sin z = \frac{1}{z^4}\sum_{k=0}^\infty \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^\infty \frac{(-1)^k z^{2k-3}}{(2k+1)!}$$

$$\frac{e^z}{z^2(1-z)} = \sum_{k=0}^\infty \frac{z^{k-2}}{k!} \cdot \sum_{\ell=0}^\infty z^\ell$$

Pick out the common exponents. Let $k - 2 + \ell = n$, then for fixed n, the value of k goes from 0 to $n+2$. The value of n goes from -2 to infinity. The sum is now

$$\sum_{n=-2}^\infty z^n \sum_{k=0}^{n+2} \frac{1}{k!}$$

The residue for the first function is $-1/6$. For the second it is $3/2$.
For $|z| > 1$ the first series is unchanged. The second one is

$$\frac{e^z}{z^2(1-z)} = \frac{e^z}{-z^3(1-1/z)} = -\sum_{k=0}^\infty \frac{z^{k-3}}{k!} \cdot \sum_{\ell=0}^\infty z^{-\ell}$$

Pick out the common exponents. Let $k-3-\ell=n$, then for fixed n, the value of k goes from $n+3$ to ∞ or from 0 to ∞ whichever is greater. The values of n go from $-\infty$ to $+\infty$.

$$-\sum_{n=-\infty}^{\infty} z^n \begin{cases} \sum_{n+3}^{\infty} 1/k! & (n\ge -3) \\ \sum_0^{\infty} 1/k! & (n<-3) \end{cases}$$

14.9 Let $z=a+iy$, then

$$\int e^{iz}dz=\int e^{ia-y}i\,dy=ie^{ia}\int_0^\infty dy\,e^{-y}=ie^{ia}$$

14.23 The only pole is at the origin, so all you need is the residue there.

$$e^{-z}z^{-n}=\sum_{k=0}^{\infty}(-1)^k\frac{z^{k-n}}{k!}$$

The coefficient of $1/z$ requires $k-n=-1$, or $k=n-1$. The integral is then $2\pi i(-1)^{n-1}/(n-1)!$.

14.30 Zero. The integrand is non-singular and odd.

14.41 At an angle that is a rational multiple of π, the function $\sum z^{n!}$ is

$$\sum_0^\infty z^{n!}=\sum_0^\infty r^{n!}e^{i\pi n!\,p/q}$$

When $n\ge q+2$, the quotient in the exponent is (an integer)$\cdot 2\pi i$. That makes the exponential $=1$. The rest is a sum $\sum^\infty r^{n!}$ and that approaches infinity as $r\to 1$. The unit circle is dense with singularities, and you can't move past it. It is called a natural boundary. And yet the function behaves so reasonably near the origin!

15.2 Fourier transform $e^{ik_0x-x^2/\sigma^2}$

$$\int dx\,e^{-ikx}e^{ik_0x-x^2/\sigma^2}=\int dx\,e^{-i(k-k_0)x}\,e^{-x^2/\sigma^2}=g(k-k_0)$$

where g is the Fourier transform of e^{-x^2/σ^2}.

15.3 For xe^{-x^2/σ^2}, start from the transform

$$g(k)=\int dx\,e^{-ikx}e^{-x^2/\sigma^2}$$

and differentiate with respect to k.

$$g'(k) = -i \int dx\, x e^{-ikx} e^{-x^2/\sigma^2}$$

The desired transform is then $ig'(k)$.

15.4 The Fourier transform[2] of f is $2\pi f(-x)$.

15.5

$$\int dy\, f(y) f(x-y) = \int_{-a}^{a} dy\, 1 \cdot \begin{cases} 1 & (-a < x-y < a) \\ 0 & \text{(elsewhere)} \end{cases}$$

It's easier to look at the inequalities if you multiply them by -1. $-a < x-y < a$ is $a > y-x > -a$ is $-a < y-x < a$. The integrand is then non-zero not just when x is within a distance $= a$ from zero, but within a distance $= a$ from x. If $x > 0$ but $x < 2a$, the overlap is from $y = x-a$ to $y = a$:

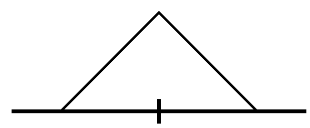
$$\int_{x-a}^{a} dy\, 1 = 2a - x$$

For x negative, the overlap is from $y = -a$ to $y = x+a$:

$$\int_{-a}^{x+a} dy\, 1 = x + 2a$$

The convolution is then

$$(f * f)(x) = \begin{cases} 2a + x & (-2a < x < 0) \\ 2a - x & (0 < x < 2a) \\ 0 & \text{(elsewhere)} \end{cases}$$



15.10 For two functions, f_1 and f_2, simply mimic the derivation as when they are the same:

$$\begin{aligned} f_1(x) &= \int \frac{dk}{2\pi}\, g_1(k)\, e^{ikx} \qquad \text{so} \\ \langle f_1, f_2 \rangle &= \int dx\, f_1^*(x) f_2(x) = \int dx \int \frac{dk}{2\pi}\, g_1^*(k)\, e^{-ikx}\, f_2(x) \\ &= \int \frac{dk}{2\pi}\, g_1^*(k) \int dx\, e^{-ikx}\, f_2(x) = \int \frac{dk}{2\pi}\, g_1^*(k)\, g_2(k) \end{aligned}$$

15.12 The critically damped SHO, and the required integral is Eq. (15.15).

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-m\omega^2 - ib\omega + k} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-m(\omega-\omega_+)(\omega-\omega_-)}$$

The only difference in this case is that the two roots are equal:

$$\omega_+ = \omega_- = -ib/2m$$

As before, if $t < t'$ the factor $e^{-i\omega(t-t')}$ is of the form $e^{+i\omega}$ and that is damped in the direction toward $+i$ in the ω-plane. The integral is then zero for $t < t'$.
In the other case, push the contour toward $-i$ and you pick up the residue at the (second order) pole.

$$G = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-m(\omega-\omega_+)^2} = -2\pi i \operatorname*{Res}_{\omega_+} \frac{e^{-i\omega(t-t')}}{-2\pi m(\omega-\omega_+)^2}$$

To get the residue,

$$e^{-i\omega(t-t')} = e^{-i(t-t')((\omega-\omega_+)+\omega_+)} = e^{-i\omega_+(t-t')}\big[1 - i(t-t')(\omega-\omega_+) + \cdots\big]$$

The coefficient of $1/(\omega-\omega_+)$ is the residue, so

$$G = \frac{i}{m} e^{-i(-ib/2m)(t-t')}\big[-i(t-t')\big] = \frac{1}{m}(t-t')e^{-bt/2m}$$

As a check, this is the limit as $\omega' \to 0$ of the equation (15.17) in the text.

15.14 The Fourier transform of $f(x) = A\big(a-|x|\big)$ [zero outside $(-a,a)$] is

$$\begin{aligned}
\int_a^a dx\, e^{-ikx} A\big(a-|x|\big) &= \int_0^a dx\, e^{-ikx} A(a-x) \quad \text{(real part, then times 2)} \\
&= Aa\frac{e^{-ika}-1}{-ik} - Ai\frac{d}{dk}\frac{e^{-ika}-1}{-ik} \\
&= Aa\frac{e^{-ika}-1}{-ik} - iA\left[\frac{-ia\, e^{-ika}}{-ik} - \frac{e^{-ika}-1}{-ik^2}\right] \\
&= Aa\left[i\frac{e^{-ika}-1}{k} - i\frac{e^{-ika}}{k}\right] - A\frac{e^{-ika}-1}{k^2} \\
\text{(real, times 2)} \quad &= 2Aa\,[0] - 2A\frac{\cos ka - 1}{k^2}
\end{aligned}$$

As a check, as $k \to 0$ this goes to $-2A\big[(1 - k^2a^2/2 + \cdots) - 1\big]/k^2 \to Aa^2$. This is the area of the triangle outlined by the original function f.
As a shrinks, the first zero of the transform move out. It is at $k = 2\pi/a$. This is a crude measure of the width of the transformed function.

15.15 The Fourier transform of $Ae^{-\alpha|x|}$ is

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, e^{-ikx} Ae^{-\alpha|x|} &= \int_{-\infty}^{0} dx\, e^{-ikx} Ae^{\alpha x} + \int_0^{\infty} dx\, e^{-ikx} Ae^{-\alpha x} \\
&= \frac{A}{\alpha - ik} - \frac{A}{-\alpha - ik} = \frac{2A\alpha}{\alpha^2 + k^2}
\end{aligned}$$

For the inverse transform, there are poles at $k = \pm i\alpha$.

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{2A\alpha}{\alpha^2 + k^2}$$

For the case $x > 0$ push the contour toward $+i$, as that's where the exponential is damped. This picks up a residue at the pole

$$2\pi i \operatorname*{Res}_{+i\alpha} e^{ikx} \frac{2A\alpha}{2\pi(k - i\alpha)(k + i\alpha)} = ie^{-\alpha x}\frac{2A\alpha}{2i\alpha} = Ae^{-kx}$$

If $x < 0$, push the contour down toward $-i$, getting the residue

$$-2\pi i \operatorname*{Res}_{-i\alpha} e^{ikx} \frac{2A\alpha}{2\pi(k - i\alpha)(k + i\alpha)} = -ie^{+\alpha x}\frac{2A\alpha}{-2i\alpha} = Ae^{+kx}$$

The loop around the contour is clockwise in the second case, requiring the second minus sign.

15.16

$$g(k) = \int_{-\infty}^{\infty} dx\, f(x)e^{-ikx} \qquad \text{then}$$

$$\int_{-\infty}^{\infty} dx\, f_1(x)e^{-ikx} = \int_{-\infty}^{\infty} dx\, f(x-a)e^{-ikx} = \int_{-\infty}^{\infty} dy\, f(y)e^{-ik(a+y)} = e^{-ika}g(k)$$

15.19 Do a Fourier transform of the equation, and integrate by parts three times.

$$\frac{d^3x}{dt^3} = F(t) \;\longrightarrow\; \int_{-\infty}^{\infty} dt\, e^{i\omega t}\frac{d^3x}{dt^3} = (-i\omega)^3\tilde{x}(\omega) = \tilde{F}(\omega)$$

Solve for the transform $\tilde{x}$ and invert.

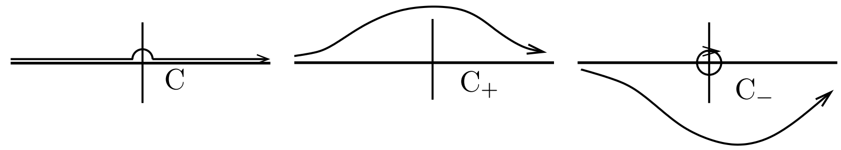
$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\frac{\tilde{F}(\omega)}{i\omega^3} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\frac{1}{i\omega^3}\int_{-\infty}^{\infty} dt'\, e^{i\omega t'}F(t')$$

Rearrange the integrals and combine the exponentials

$$x(t) = \int_{-\infty}^{\infty} dt'\, F(t')\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{1}{i\omega^3}e^{-i\omega(t-t')}$$

To do the ω integral, treat it as a contour integral and modify the contour as stated so that it doesn't go straight through the pole at zero. Instead the contour goes slightly above the pole.

$$\int_C d\omega\, \frac{1}{\omega^3}e^{-i\omega(t-t')}$$



If $t < t'$ the you can push the contour up toward $+i\infty$ (C_+) and the exponential kills it. In the reverse case you push the contour toward $-i\infty$ and the exponential kills the contour over the large arc, leaving only the residue at the origin (C_-).

$$\int_{C_-} = -2\pi i \operatorname*{Res}_{\omega=0} \frac{1}{\omega^3}\big[1 - i\omega(t-t') - \omega^2(t-t')^2/2 + \cdots\big] = -2\pi i\big[-(t-t')^2/2\big]$$

Put this back into the integral for $x(t)$ and you have

$$x(t) = \frac{1}{2}\int_{-\infty}^{t} dt'\, F(t')(t-t')^2$$

When you pick an example, you can't use anything *quite* as simple as a constant or a small power, because the integral won't converge. You can however try a constant on an interval.

$$F(t) = \begin{cases} 1 & (-t_0 < t < t_0) \\ 0 & \text{(elsewhere)} \end{cases} \;\rightarrow\; x(t) = \frac{1}{2}\int_{-t_0}^{t} dt'\, 1(t-t')^2 = \frac{1}{6}(t+t_0)^3$$

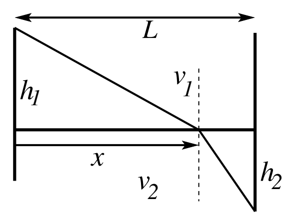
This applies only to the interval $-t_0 < t < t_0$. It is zero for smaller values of t, and for $t > t_0$ it is

$$x(t) = \frac{1}{2}\int_{-t_0}^{t_0} dt'\, 1(t-t')^2 = \frac{1}{6}\big[-(t-t_0)^3 + (t+t_0)^3\big] = \frac{1}{6}\big[6t_0t^2 + 2t_0^3\big]$$

You can verify that x and its first and second derivatives are continuous at the point t_0.

16.1 The two straight lines represent the shortest time paths when the speed is constant. The total travel time for the two speeds is

$$T = \frac{1}{v_1}\sqrt{h_1^2 + x^2} + \frac{1}{v_2}\sqrt{h_2^2 + (L-x)^2}$$



To minimize this time, vary x, setting the derivative to zero.

$$\frac{dT}{dx} = \frac{1}{v_1}\frac{x}{\sqrt{h_1^2 + x^2}} - \frac{1}{v_2}\frac{L-x}{\sqrt{h_2^2 + (L-x)^2}} = 0$$

Reinterpret the results in terms of the angles from the normal, and

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

This is Snell's Law for refraction.

16.2 A point moving on a circle centered at the origin is $x = -R\sin\omega t$ and $y = -R\cos\omega t$. Now raise it so that the circle touches the x-axis and cause to move right so that the velocity of the center will be $R\omega$. The latter will mean that when the moving point is at the bottom of the circle its total velocity will be $+R\omega - R\omega = 0$.

$$x = R\omega t - R\sin\omega t, \qquad y = R - R\cos\omega t$$

Let $\theta = \omega t$ and then eliminate it.

$$\theta = \cos^{-1}\big[(y-R)/R\big], \qquad \text{then} \qquad x = R\cos^{-1}\left(\frac{y-R}{R}\right) + R\sqrt{1-(y-R)^2/R^2}$$
$$= R\cos^{-1}\left(\frac{y-R}{R}\right) + \sqrt{R^2-2Ry}$$

16.6 The optical path over a hot road, but with a different independent variable.

$$\int n\,d\ell = \int f(y)\sqrt{1+y'^2}dx$$

This integrand does not contain the independent variable x. That makes it susceptible to the already partly integrated form of the Euler-Lagrange differential equation

$$y'\frac{\partial F}{\partial y'} - F = C = y'\frac{f(y)y'}{\sqrt{1+y'^2}} - f(y)\sqrt{1+y'^2} = \frac{-f(y)}{\sqrt{1+y'^2}}$$

Rearrange this as

$$C^2\big(1+y'^2\big) = f^2, \qquad \text{or} \qquad y' = \frac{dy}{dx} = \sqrt{(f^2/C^2)-1}$$

This is identical to the differential equation found before for this problem, Eq. (16.24), so it has the same solution.

16.13 "Develop the cylinder." That means to slice the cylinder along a line such as that parallel to the z-axis and then lay the result down in a plane. You can do this because the cylinder is really flat. The shortest distance is a straight line in the plane, translating to a helix on the cylinder.
OR, write $S = \int d\ell = \int d\theta\,\sqrt{R^2+(dz/d\theta)^2}$. Now use the Euler-Lagrange equation $\delta S/\delta x = 0$. This is $d^2x/d\theta^2 = 0$.

16.14 Use cylindrical coordinates, and the radius is a function of z. Let the height be $2h$, and the area is

$$A = \int 2\pi r\sqrt{dz^2+dr^2} = 2\pi\int_{-h}^{+h} r\,dz\sqrt{1+(dr/dz)^2} = 2\pi\int_{-h}^{+h} dz\,F(r,r')$$

The area is a functional of r: $A[r]$. Set the functional derivative to zero. Notice first that this is a case for which the integrand is independent of z, making it appropriate to use equation (16.21).

$$F - r'\frac{\partial F}{\partial r'} = C = r\sqrt{1+r'^2} - r'\cdot\frac{rr'}{\sqrt{1+r'^2}} = \frac{r}{\sqrt{1+r'^2}}$$

Rearrange this, solving for r'.

$$C^2\big(1+r'^2\big) = r^2, \qquad \text{then} \qquad r' = \sqrt{r^2/C^2-1}, \qquad \text{and} \qquad \frac{dr}{\sqrt{r^2-C^2}} = dz/C$$

Substitute $r = C\cosh\theta$, then

$$\frac{C\sinh\theta d\theta}{C\sinh\theta} = dz/C, \qquad \text{giving} \qquad \theta = z/C + D \qquad \text{and} \qquad r = C\cosh\big[(z/C)+D\big]$$

With the boundary condition that this is even in x, you have $D = 0$. Then let $r(h) = R$, the common radius of the rings.

$$R = C\cosh(h/C)$$

This is an equation for the parameter C. It will not have a solution if h/R is too large. To see why this is so, let $\alpha = h/C$ then $\alpha R/h = \cosh(\alpha)$ Plot the two sides of this equation versus α, and you will see that if R/h is too small the two curves don't intersect. The limit occurs when the two curves are tangent and is $R/h = 1.509$. In that limit, the radius of the surface at $z = 0$ is $0.55R$.

16.17 For the simple harmonic oscillator Lagrangian, use an explicit variation to calculate everything.

$$S[x] = \int_0^T dt\,\big[m\dot{x}^2/2 - m\omega^2x^2/2\big]$$

$$\begin{aligned}\delta S = S[x+\delta x] - S[x] &= \int_0^T dt\,\big[m\dot{x}\delta\dot{x} + \delta\dot{x}^2/2 - m\omega^2 x\delta x - m\omega^2\delta x^2/2\big] \\ &= m\dot{x}\delta x\Big|_0^T + \int_0^T dt\,\big[-m\ddot{x}\delta x + \delta\dot{x}^2/2 - m\omega^2 x\delta x - m\omega^2\delta x^2/2\big]\end{aligned}$$

With the usual assumption that the endpoints are fixed, δx vanishes at 0 and T. That kills the surface terms, then for the first order variation to vanish you have $\ddot{x} + \omega^2 x = 0$, the harmonic oscillator. Now pick the explicit $\delta x = \epsilon\sin(n\pi t/T)$ to evaluate the second order terms. This satisfies the boundary conditions, and

$$\begin{aligned}\delta S &= \int_0^T dt\,\Big[\big(\epsilon(n\pi/T)\sin(n\pi t/T)\big)^2/2 - m\omega^2\big(\epsilon\sin(n\pi t/T)\big)^2/2\Big] \\ &= \big(\epsilon^2/4\big)\big((n\pi/T)^2 - \omega^2\big) \\ &= \big(\epsilon^2/4T^2\big)\big((n\pi)^2 - (\omega T)^2\big)\end{aligned}$$

If $\omega T < \pi$ this is positive for all the positive integers n. It is a minimum. If $2\pi > \omega T > 1\pi$, then for the $n = 1$ variation the second order term is negative, and S is a maximum with respect to changes in this direction. It is a minimum with respect to changes in the other ($n \geq 2$) directions. This solution is then a saddle point and not a minimum.
What is special about this value of ωT? It is a focus. For the time $T = \pi/\omega$, all the initial conditions on the differential equation starting at $x(0) = 0$ take you to the same point $x(T) = 0$. In a lens that describes a focus, and the same term is used here.
This phenomenon is quite general; the presence of a focus changed a minimum to a saddle point. See problems 2.35 and 2.39.

16.26 Use the same geometry as in the figure accompanying Eq. (16.49), with p being the distance from the source to the near side of the lens. q is the distance from the far side.

$$t_1 = \frac{1}{c}(p+q) + \frac{n(0)}{c}t, \qquad t_2 = \frac{1}{c}\sqrt{p^2+r^2} + \frac{1}{c}\sqrt{q^2+r^2} + \frac{n(r)}{c}t$$

Equate these two times and assume that $r \ll p,\, q$. Drop the common factor c.

$$\begin{aligned}(p+q) + n(0)t &= p\left(1 + \frac{r^2}{2p^2}\right) + q\left(1 + \frac{r^2}{2q^2}\right) + n(r)t \\ n(0)t &= \frac{r^2}{2p} + \frac{r^2}{2q} + n(r)t \\ n(r) &= n(0) - \frac{r^2}{2t}\left(\frac{1}{p} + \frac{1}{q}\right)\end{aligned}$$

The focus obeys $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$, so this becomes $n(r)=n(0)-r^2/2ft$.

17.1 For the Gaussian distribution,

$$\text{Mean:}\quad \int dg\, g A e^{-B(g-g_0)^2} = \int dg\,(g-g_0+g_0)Ae^{-B(g-g_0)^2} = g_0$$

Note the normalization: $\int Ae^{-B(g-g_0)^2} = A\sqrt{\pi/B}=1$.

$$\begin{aligned}\text{Variance:}\quad \sigma^2 &= \int dg\,(g-g_0)^2 Ae^{-B(g-g_0)^2} = -A\frac{d}{dB}\int dg\, e^{-B(g-g_0)^2}\\ &= -A\frac{d}{dB}\sqrt{\frac{\pi}{B}} = -A\frac{-1}{2}\sqrt{\frac{\pi}{B^3}} = \frac{1}{2B}\end{aligned}$$

$$\text{Skewness:}\quad \int dg\,(g-g_0)^3 Ae^{-B(g-g_0)^2} = 0$$

$$\begin{aligned}\text{Kurtosis excess:}\quad & -3+\frac{1}{\sigma^4}\int dg\,(g-g_0)^4 Ae^{-B(g-g_0)^2}\\ &= -3 + A\frac{1}{\sigma^4}\frac{d^2}{dB^2}\int dg\, e^{-B(g-g_0)^2}\\ &= -3 + A\frac{1}{\sigma^4}\frac{d^2}{dB^2}\sqrt{\frac{\pi}{B}} = -3 + A\frac{1}{\sigma^4}\frac{3}{4}\sqrt{\frac{\pi}{B^5}}\\ &= -3 + A\cdot 4B^2\cdot\frac{3}{4}\sqrt{\frac{\pi}{B^5}} = -3+3=0\end{aligned}$$

17.2 For the flat distribution, $f(g)=C$ for $(0<g<g_\mathsf{m})$

$$\text{Mean:}\quad \int_0^{g_\mathsf{m}} dg\, g\, C = \int_0^{g_\mathsf{m}} dg\,\big((g-{}^1\!/\!_2\, g_\mathsf{m}) + {}^1\!/\!_2\, g_\mathsf{m}\big)C = {}^1\!/\!_2\, g_\mathsf{m}$$

Note the normalization: $\int dg\, C = g_\mathsf{m}C = 1$. Also, define the center point as $g_\mathsf{c}=g_\mathsf{m}/2$.

$$\begin{aligned}\text{Variance:}\quad \sigma^2 &= \int dg\,(g-g_\mathsf{c})^2 C = C\int_{-g_\mathsf{c}}^{+g_\mathsf{c}} dg'\, g'^2\\ &= 2Cg_\mathsf{c}^3/3 = g_\mathsf{c}^2/3 = g_\mathsf{m}^2/12\end{aligned}$$

$$\text{Skewness:}\quad \int dg\,(g-g_\mathsf{c})^3 C = 0$$

$$\begin{aligned}\text{Kurtosis excess:}\quad & -3+\frac{1}{\sigma^4}\int dg\,(g-g_\mathsf{c})^4 C = -3+\frac{1}{\sigma^4}C\int_{-g_\mathsf{c}}^{+g_\mathsf{c}} dg'\, g'^4\\ &= -3+\frac{9}{g_\mathsf{c}^4}C2g_\mathsf{c}^5/5 = -3+\frac{9}{g_\mathsf{c}^4}g_\mathsf{c}^4/5 = -3+\frac{9}{5} = -\frac{6}{5}\end{aligned}$$

17.5 The reciprocal of its argument.

17.9 The step whose derivative is $\delta_n(x) = \sqrt{n/\pi}e^{-nx^2}$ is

$$\theta_n(x) = \int_{-\infty}^{x} dx' \sqrt{\frac{n}{\pi}}\, e^{-nx'^2} = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{x\sqrt{n}} \frac{dt}{\sqrt{n}}\, e^{-t^2} = \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{0} + \int_{0}^{x\sqrt{n}} \right] dt\, e^{-t^2}$$

Recall the definition of the error function, Eq. (1.11), and that it goes from -1 to $+1$ over all x, to see that this is

$$\theta_n(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(x\sqrt{n}\right)\right]$$

17.11 Change variables in Eq. (17.30) to $x' = x - y$:

$$\frac{d^2g}{dx^2} - k^2 g = \delta(x-y) \;\longrightarrow\; \frac{d^2g}{dx'^2} - k^2 g = \delta(x')$$

This has a discontinuity at zero, and everything in the equation is even in the x' variable. When you write the solution for $x' > 0$ as $g(x') = A + Be^{-kx'}$ and then say that this should go to zero for large x' you eliminate the constant A. Similarly for negative x' the B constant must be zero to keep the function g bounded. Continuity of the function at the origin then implies that

$$g(x') = \begin{cases} Be^{kx'} & (x' < 0) \\ Be^{-kx'} & (x > 0) \end{cases} \quad = Be^{-k|x'|}$$

The discontinuity in g' is $g'(0+) - g'(0-) = -2kB$. The differential equation says that g'' has a delta-function at $x' = 0$, so the integral of g'' from just below to just above zero is

$$\int_{0-}^{0+} g''(x')\, dx' = g'(0+) - g'(0-) = \int_{0-}^{0+} \delta(x')\, dx' = 1$$

This implies $-2kB = 1$, so

$$g(x') = -\frac{1}{2k}e^{-k|x'|} = -\frac{1}{2k}e^{-k|x-y|}$$

and this agrees with the longer derivation in the text.